MCV4U: Calculus & Vectors

Vertical, Horizontal and Oblique Asymptotes

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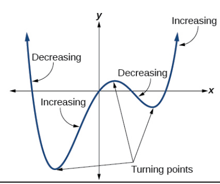


Critical Points

Recap

Determine any critical values for $y = \frac{e^x}{\sqrt{x}}$.

The derivative is $\frac{dy}{dx} = \frac{e^x\sqrt{x} - e^x\left(\frac{1}{2\sqrt{x}}\right)}{x}$, which simplifies to $\frac{dy}{dx} = \frac{e^x(2x-1)}{2\sqrt{x^3}}$.

Since $e^x \neq 0$, the only critical point occurs when $2x - 1 = 0$, or $x = \frac{1}{2}$.

When $x = \frac{1}{2}$, the critical value is $y = \frac{e^{1/2}}{\sqrt{\frac{1}{2}}} = \sqrt{2e}$.

Vertical Asymptotes

While polynomial functions do not have any *vertical asymptotes*, they often occur in rational functions.

For rational functions involving polynomials, there will be a vertical asymptote at $x = k$ if $x - k$ is a factor of the denominator, provided it is not also a factor of the numerator.

If $x - k$ is a factor of both the numerator and the denominator, it is a point discontinuity (hole) instead.

Other functions may contain vertical asymptotes at values where the denominator equates to zero. For example, $y = \tan x$ has vertical asymptotes for all values of x where $\cos x = 0$, since $\tan x = \frac{\sin x}{\cos x}$.

Vertical Asymptotes

Example

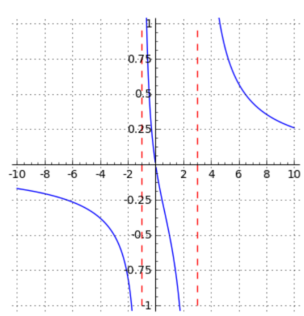
Determine any vertical asymptotes for $f(x) = \frac{2x}{x^2 - 2x - 3}$.

Factoring the denominator, we can rewrite the function as $f(x) = \frac{2x}{(x-3)(x+1)}$.

Therefore, there are vertical asymptotes at $x = 3$ and $x = -1$.

Vertical Asymptotes

A graph of y shows how the function approaches the asymptotes at $x = 3$ and $x = -1$.



Vertical Asymptotes

Example

Determine any vertical asymptotes for $f(x) = \frac{x-1}{x^3 - x^2 + 2x - 2}$.

Factor the denominator by grouping.

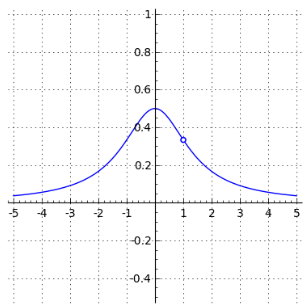
$$\begin{aligned} x^3 - x^2 + 2x - 2 &= x^2(x-1) + 2(x-1) \\ &= (x^2+2)(x-1) \end{aligned}$$

Thus, we can rewrite the function as $f(x) = \frac{1}{x^2+2}$, $x \neq 1$.

Since $f(1) = \frac{1}{3}$, there is a point discontinuity at $\left(1, \frac{1}{3}\right)$.

Vertical Asymptotes

Since $x^2 + 2 \geq 2$, the denominator is never equal to zero. Thus, there are no vertical asymptotes.



Horizontal Asymptotes

A *horizontal asymptote* is a value that a function approaches as $x \to \infty$ or $x \to -\infty$.

Formally, if $\lim_{x\to\pm\infty} f(x) = L$, then $y = L$ is a horizontal asymptote for $f(x)$.

In previous courses, you have probably determined the equations of horizontal asymptotes by dividing polynomial terms by the highest power, then observing what happens as either $x \to \infty$ or $x \to -\infty$.

This is the same technique used here, but formalized using limit notation.

Horizontal Asymptotes

Example

Determine any horizontal asymptotes for $y = \dfrac{4x^2 - 3x}{2x^2 + 5}$.

Factor the highest power, x^2, from each term.

$$\frac{4x^2 - 3x}{2x^2 + 5} = \frac{x^2\left(4 - \frac{3}{x}\right)}{x^2\left(2 + \frac{5}{x^2}\right)} = \frac{4 - \frac{3}{x}}{2 + \frac{5}{x^2}}$$

Horizontal Asymptotes

Evaluate the limits as $x \to \infty$ and $x \to -\infty$.

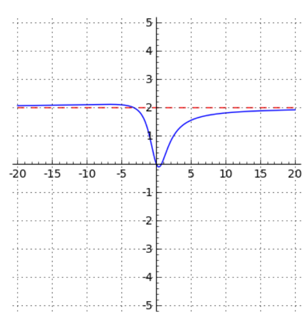
$$\lim_{x\to\infty}\left(\frac{4 - \frac{3}{x}}{2 + \frac{5}{x^2}}\right) = \frac{4-0}{2+0} = 2 \qquad \lim_{x\to-\infty}\left(\frac{4 - \frac{3}{x}}{2 + \frac{5}{x^2}}\right) = \frac{4-0}{2+0} = 2$$

In this case, the line $y = 2$ is a horizontal asymptote as $x \to \infty$ and as $x \to -\infty$.

This is not always the case. Some functions may have different asymptotes as x gets very large, or very small.

Horizontal Asymptotes

A graph of y shows the function approaching $y = 2$ as $x \to \infty$ and as $x \to -\infty$.



Horizontal Asymptotes

Example

Determine any horizontal asymptotes for $y = \dfrac{x+2}{\sqrt{9x^2+1}}$.

The square root complicates matters here, so we need to approach it with an analytical perspective.

As $x \to \infty$, $9x^2$ will have a much greater effect than the $+1$ in the denominator. When x is sufficiently large, the 1 will be almost insignificant.

Therefore, the denominator may be approximated by $\sqrt{9x^2}$, or $3x$, so we can use x as the highest power.

This gives us $y = \dfrac{x\left(1 + \frac{2}{x}\right)}{x\left(\frac{\sqrt{9x^2+1}}{x}\right)}$, or $y = \dfrac{1 + \frac{2}{x}}{\frac{\sqrt{9x^2+1}}{x}}$.

Horizontal Asymptotes

The square root continues to be an issue in the denominator, so we need to simplify further.

Since x is positive when $x \to \infty$, we can use the fact that $x = \sqrt{x^2}$. Similarly, x is negative when $x \to -\infty$, so $x = -\sqrt{x^2}$.

$$\frac{1+\frac{2}{x}}{\frac{\sqrt{9x^2+1}}{\sqrt{x^2}}} = \frac{1+\frac{2}{x}}{\sqrt{\frac{9x^2+1}{x^2}}} = \frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}} \qquad \frac{1+\frac{2}{x}}{\frac{\sqrt{9x^2+1}}{-\sqrt{x^2}}} = \frac{1+\frac{2}{x}}{-\sqrt{\frac{9x^2+1}{x^2}}} = \frac{1+\frac{2}{x}}{-\sqrt{9+\frac{1}{x^2}}}$$

Horizontal Asymptotes

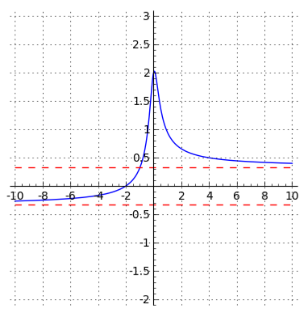
Finally, we can evaluate the limits as $x \to \infty$ and $x \to -\infty$.

$$\lim_{x\to\infty} \frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3} \qquad \lim_{x\to-\infty} \frac{1+\frac{2}{x}}{-\sqrt{9+\frac{1}{x^2}}} = \frac{1+0}{-\sqrt{9+0}} = -\frac{1}{3}$$

Therefore, there are two horizontal asymptotes for the function. As $x \to \infty$, there is a horizontal asymptote at $y = \frac{1}{3}$, and as $x \to -\infty$, there is one at $y = -\frac{1}{3}$.

Horizontal Asymptotes

A graph of y shows both horizontal asymptotes.



Oblique Asymptotes

An *oblique asymptote*, or *slant asymptote*, is a linear asymptote that is neither vertical nor horizontal.

For rational functions involving polynomials, oblique asymptotes occur when the numerator has a degree one greater than the denominator.

For example, the rational function $y = \frac{x^2+3x}{x-2}$ has an oblique asymptote with equation $y = x + 5$, whereas the rational function $y = \frac{x}{x^2-4}$ does not have an oblique asymptote.

Rational functions involving polynomials may contain a horizontal asymptote or an oblique asymptote, but not both.

The equation of an oblique asymptote may be found using long or synthetic division.

Oblique Asymptotes

Example

Determine the equation of any oblique asymptotes for $f(x) = \frac{x^2+5x-4}{x+3}$.

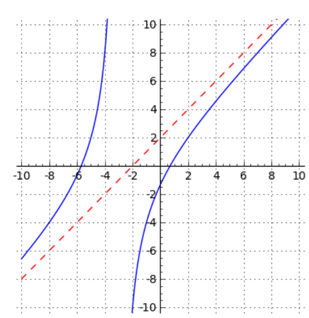
Use $x = -3$ and synthetic division to find the quotient, which is the equation of the asymptote.

	1	5	−4
−3		−3	−6
	1	2	−10

Therefore, $f(x)$ has an oblique asymptote with equation $y = x + 2$.

Oblique Asymptotes

A graph of $f(x)$ is below.



Other Asymptotes

While the asymptotes covered in this course are limited to vertical, horizontal and oblique, an asymptote may assume the shape of any function.

For example, a rational function with a numerator of order 3 and a denominator of order 1 will have a *parabolic asymptote*, while a quintic numerator and a quadratic denominator will result in a *cubic asymptote*.

In general, a rational function with a polynomial or order n and a denominator of order m, with $n > m$, will have a polynomial asymptote with order $n - m$.

Other Asymptotes

Example

Determine the equation of the parabolic asymptote to $y = \dfrac{x^3 + 4x^2 - 2x + 1}{x - 2}$.

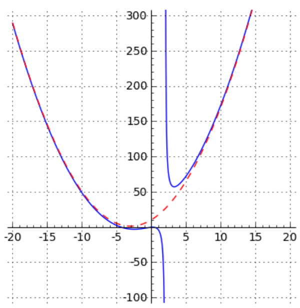
Use synthetic division to determine the equation.

	1	4	−2	1
2		2	12	20
	1	6	10	21

The equation of the parabolic asymptote is $y = x^2 + 6x + 10$, or $y = (x + 3)^2 + 1$.

Other Asymptotes

A graph of y shows how the function approaches the asymptote.



Questions?