MCR3U: Functions

Solving Exponential Equations

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Solving Exponential Equations

As we will see shortly, many phenomena can be modelled using exponential equations.

While it is easy to evaluate exponential equations (i.e. plugging in values for x), we often need to work backwards and solve for x.

In some cases, this can be done easily by inspection.

In other cases, we can try to find a common base to simplify all powers in the equation, then use the laws of exponents solve it.

Change of Base

Example

Solve $2^x = 64$.

Since $2^6 = 64$, the solution is $x = 6$. In this case, it is necessary to recognize 64 as a power of 2, or to use "guess and check" to determine the solution.

Example

Solve $9^x = 27$.

Rewriting both 9^x and 27 as powers of 3, we obtain $3^{2x} = 3^3$.

Since the base is the same on both sides, we can "cancel" the base and deal only with the exponents. Thus, the solution occurs when $2x = 3$, or $x = \frac{3}{2}$.

Change of Base

Example

Solve $25^{3x} = 125^{2x}$.

Using a common base of 5, $\left(5^2\right)^{3x} = \left(5^3\right)^{2x}$, or $5^{6x} = 5^{6x}$. This is true for all values of x.

To test this, substitute any value, like $x = 3$, into the original equations.

$$\begin{aligned} 25^{3\times3} &= 25^9 & \qquad 125^{2\times3} &= 125^6 \\ &\approx 3.8\times10^{12} & &\approx 3.8\times10^{12} \end{aligned}$$

Change of Base

Example

Solve $4^{5x-1} = 8^{2x+3}$.

Use a common base of 2, and use the distributive property in the exponents.

$$\begin{aligned} \left(2^2\right)^{5x-1} &= \left(2^3\right)^{2x+3} \\ 2^{10x-2} &= 2^{6x+9} \\ 10x-2 &= 6x+9 \\ 4x &= 11 \\ x &= \tfrac{11}{4} \end{aligned}$$

The solution is $x = \frac{11}{4}$.

Change of Base

Example

Solve $\dfrac{9^{x-3}}{3^{4x+1}} = 81^{x+2}$.

Use a common base of 3, then use the quotient rule for exponents.

$$\begin{aligned} \frac{\left(3^2\right)^{x-3}}{3^{4x+1}} &= \left(3^4\right)^{x+2} \\ \frac{3^{2x-6}}{3^{4x+1}} &= 3^{4x+8} \\ 3^{-2x-7} &= 3^{4x+8} \\ -2x-7 &= 4x+8 \\ x &= -\tfrac{5}{2} \end{aligned}$$

The solution is $x = -\frac{5}{2}$.

Factoring

Another method of solving equations involving exponents is by using factoring techniques.

In many cases, using some form of substitution (e.g. letting $k = 5^x$) may make this process clearer.

Each factor must be checked to determine whether it contains a solution to the equation and, depending on the number of factors, there may be more than one solution.

Factoring

Example

Solve $2^{2x} - 2^x = 0$.

Recall that $2^{2x} = (2^x)^2$. Thus, we can rewrite the equation as $(2^x)^2 - 2^x = 0$.

If $k = 2^x$, then the expression becomes $k^2 - k = 0$, which factors as $k(k-1) = 0$.

If $k = 0$, then $2^x = 0$. This has no real solution.

If $k - 1 = 0$, then $2^x - 1 = 0$, or $2^x = 1$. This has the solution $x = 0$.

Therefore, there is one solution to the equation.

Factoring

Example

Solve $3 \cdot 3^{2x} - 28 \cdot 3^x + 9 = 0$.

If $k = 3^x$, then the equation becomes $3k^2 - 28k + 9 = 0$.

This factors as $(3k-1)(k-9) = 0$, so either $3 \cdot 3^x - 1 = 0$ or $3^x - 9 = 0$.

If $3 \cdot 3^x - 1 = 0$, then $3^x = \frac{1}{3}$. This has the solution $x = -1$.

If $3^x - 9 = 0$, then $3^x = 9$. This has the solution $x = 2$.

Therefore, there are two solutions to the equation.

Other Methods of Solving

When the previous techniques fail, there are two other methods for solving exponential equations.

- using "guess and check"
- using *logarithms*

The second of these options, while relatively straightforward, is beyond the scope of this course and is covered in MHF4U instead.

Other Methods of Solving

Example

Solve $5^{3x} = 30$.

Since there is no common base to represent both 5 and 30, we can only substitute values into the expression and try to get as close to 30 as possible.

Since 30 is closer to 5^2 than it is to 5^3, the exponent should be closer to 2 than 3.

If $3x = 2$, then $x = \frac{2}{3}$, so we should try substituting values slightly above $\frac{2}{3}$.

Indeed, when $x = 0.7$, then $5^{3 \times 0.7} \approx 29.4$.

When $x = 0.705$, then $5^{3 \times 0.705} \approx 30.1$. This is close enough for our purposes.

Questions?