MCR3U: Functions

Recursive Sequences

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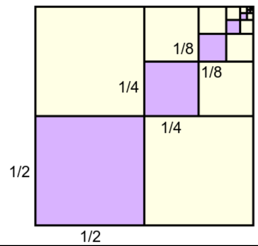


Recursive Sequences

Consider the following sequence: $1, 1, 2, 3, 5, 8, \ldots$. What are the next three terms?

The next three terms are 13, 21 and 34.

While the sequence is neither arithmetic (no common difference) nor geometric (no common ratio), there is a predictable pattern.

Each term, after the first two, is the sum of the previous two terms.

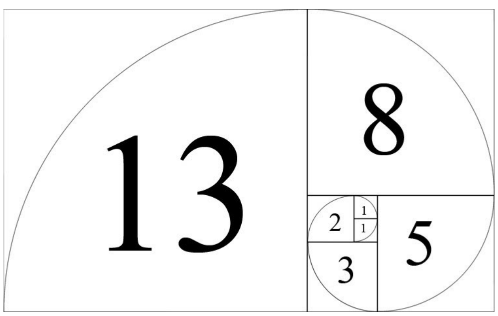
For example, $t_4 = t_3 + t_2$, or $3 = 2 + 1$.

This sequence is known as the *Fibonacci sequence*, and is an example of a *recursive sequence*.

It is often cited as a commonly sequence occurring in nature.

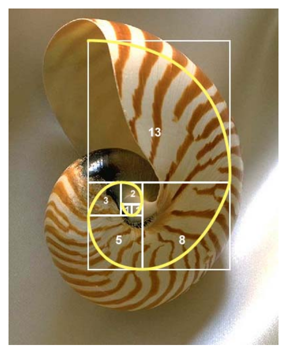
Recursive Sequences

The Fibonacci spiral:



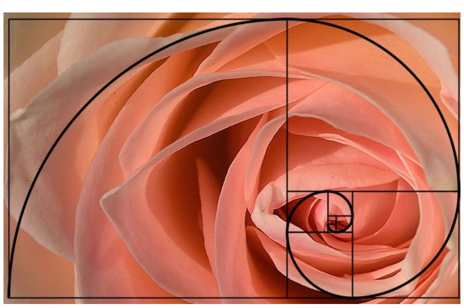
Recursive Sequences

A nautilus shell, roughly in the shape of the Fibonacci spiral.



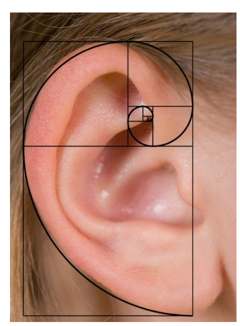
Recursive Sequences

Maybe.



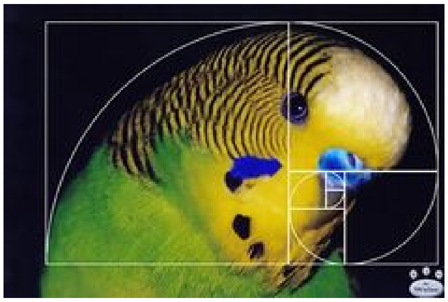
Recursive Sequences

Not really.



Recursive Sequences

Probably not.



Recursive Sequences

The general term of the Fibonacci sequence is $t_n = t_{n-1} + t_{n-2}; t_1 = 1, t_2 = 1$.

There are two parts to this formula:

- a *recurrence formula*, which describes how the previous term(s) are used to create subsequent terms, and
- one or more *initial conditions* (or *base cases*) that define the values of one or more terms.

The Fibonacci sequence has two initial conditions, since the recurrence formula involves two previous terms.

Recursive Sequences

To find the value of, say, the 20th term, we would first need to work out the values of the 19th and 18th terms.

Calculating the value of the 18th term would require the values of the 17th and 16th terms.

The 16th term depends on the values of the 15th and 14th terms, and so forth.

Sometimes it is possible to derive an *explicit formula* for the general term of a recursive sequence, but not in all cases.

In the case of the Fibonacci sequence, the explicit formula is $t_n = \dfrac{\Phi^n - (1-\Phi)^n}{\sqrt{5}}$, where $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $n \geq 1$.

While this formula is rather complicated, other recursive sequences can have much simpler explicit formulae.

Recursive Sequences

We can generalize some characteristics of all recursive sequences.

Recursive Sequences

A sequence is recursive if it is defined as a function of previous terms. All recursive sequences must specify both a recursive function, and a set of initial conditions. A recursive sequence may have an explicit formula, or it may not.

The recursive function is necessary because, without it, we would not know how to use the previous terms to define subsequent values.

The initial conditions are necessary because, without them, we would not know where to begin the sequence.

Recursive Sequences

Example

Determine the next three terms in the recursive sequence $t_n = t_{n-1} + 7; t_1 = 3$.

Calculating terms t_2, t_3 and t_4, we obtain:

- $t_2 = t_1 + 7 = 3 + 7 = 10$
- $t_3 = t_2 + 7 = 10 + 7 = 17$
- $t_4 = t_3 + 7 = 17 + 7 = 24$

Note that this sequence is arithmetic, with general term $t_n = 3 + (n-1)(7)$, or $t_n = 7n - 4$.

Recursive Formula for a Arithmetic Sequence

An arithmetic sequence with general term $t_n = t_1 + (n-1)d$ has a recursive formula $t_n = t_{n-1} + d$.

Recursive Sequences

Example

Determine the next three terms in the recursive sequence $t_n = 3 \cdot t_{n-1}; t_1 = 5$.

Calculating terms t_2, t_3 and t_4, we obtain:

- $t_2 = 3 \cdot t_1 = 3 \cdot 5 = 15$
- $t_3 = 3 \cdot t_2 = 3 \cdot 15 = 45$
- $t_4 = 3 \cdot t_3 = 3 \cdot 45 = 135$

Note that this sequence is geometric, with general term $t_n = 5 \cdot 3^{n-1}$.

Recursive Formula for a Geometric Sequence

A geometric sequence with general term $t_n = t_1 \cdot r^{n-1}$ has a recursive formula $t_n = r \cdot t_{n-1}$.

Recursive Sequences

Example

Determine an explicit formula for the sequence $t_n = t_{n-1} - 8; t_1 = 29$.

This sequence is arithmetic, with a common difference $d = -8$ and an initial term $t_1 = 29$.

$$t_n = 29 + (n-1)(-8)$$
$$t_n = 37 - 8n$$

Sometimes a recursive sequence requires more in-depth analysis in order to determine a formula.

Recursive Sequences

Example

Determine a recursive formula for the sequence $16, 21, 31, 46, \ldots$.

The table below summarizes how the value of each term is made from the previous term.

n	t_n	change	rep.
1	16	$+0$	$16 + 5 \times 0$
2	21	$+5$	$16 + 5 \times 1$
3	31	$+10$	$16 + 5 \times 2$
4	46	$+15$	$16 + 5 \times 3$

Thus, a recursive formula is $t_n = t_{n-1} + 5(n-1); t_1 = 16$.

Recursive Sequences

Example

Determine a recursive formula for the sequence $2, 3, 6, 18, 108, \ldots$.

At first it may not be obvious what is happening, since the sequence is neither arithmetic nor geometric.

Since $2 \times 3 = 6$, $3 \times 6 = 18$ and $6 \times 18 = 108$, the sequence is generated by multiplying the two previous terms.

This requires two initial cases, $t_1 = 2$ and $t_2 = 3$.

Therefore, a recursive formula for the sequence is $t_n = t_{n-1} \times t_{n-2}; t_1 = 2, t_2 = 3$.

Recursive Sequences