MCR3U: Functions

Mortgages

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Annuities

Recap

How much needs to be invested in an account, paying 6.5%/a interest, compounded bi-monthly, to provide for withdrawals of \$3 000 every two months for 15 years?

$$P = 3\,000 \cdot \frac{1-\left(1+\frac{0.065}{6}\right)^{-15\times 6}}{\frac{0.065}{6}}$$
$$\approx \$171\,920.68$$

Just under \$172 000 needs to be deposited.

Effective Interest Rates

Up to this point, all questions have involved situations where the compounding frequency is the same as the payment/withdrawal frequency.

In some cases, these do not match up. For instance, an account might compound interest monthly, but amounts may be deposited on a weekly basis.

To account for this mismatch, we must calculate an *effective rate* or *equivalent rate*.

Banks and credit card companies often do this, stating an annual rate of interest (compounded monthly) and an effective annual interest rate.

For instance, a credit card that charges 18%/a interest, compounded monthly, will have an effective annual interest rate of around 19.56%. Where did this number come from?

Effective Interest Rates

Consider the case where \$1 is invested at 18%/a interest, compounded monthly.

At the end of the first compounding period, the investment will be worth $1\left(1+\frac{0.18}{12}\right)$, after the second $1\left(1+\frac{0.18}{12}\right)^2$, after the third $1\left(1+\frac{0.18}{12}\right)^3$, and so on, until the end of the last compounding period where it is worth $1\left(1+\frac{0.18}{12}\right)^{12}$.

Thus, at the end of one year, the \$1 will grow to a value of $1\left(1+\frac{0.18}{12}\right)^{12} \approx \$1.1956$, an increase of around 19.56%.

Effective Interest Rates

What about the case where an effective annual interest rate is to be expressed as a monthly rate?

For example, a bank may offer a loan with an effective annual interest rate of 7%. What is the monthly rate?

This time, \$1 grows to 1.07 in 12 months according to the equation $1(1+m)^{12} = 1.07$, where m is some monthly rate.

$$\begin{aligned} 1(1+m)^{12} &= 1.07 \\ 1+m &= \sqrt[12]{1.07} \\ m &= \sqrt[12]{1.07}-1 \\ &\approx 0.005\,654\,145 \end{aligned}$$

So the monthly rate is approximately 0.565%.

Effective Interest Rates

Finally, what about a loan that charges 6%/a, compounded semi-annually, but is paid off in monthly instalments?

If the loan is compounded semi-annually, then \$1 will grow to $1+\frac{0.06}{2} = \$1.03$ in 6 months. This is the semi-annual rate.

In those 6 months, the \$1 grows to that same amount using some monthly rate m. Thus, $1(1+m)^6 = 1.03$.

Solving this gives us the monthly rate.

$$\begin{aligned} 1(1+m)^{6} &= 1.03 \\ 1+m &= \sqrt[6]{1.03} \\ m &= \sqrt[6]{1.03}-1 \\ &\approx 0.004\,938\,622 \end{aligned}$$

The monthly rate is about 0.494%.

Effective Interest Rates

Example

Calculate the effective annual interest rate and the equivalent monthly rate for a 5%/a interest rate, compounded semi-annually.

\$1 invested at 5%/a, compounded semi-annually, grows to $1 + \frac{0.05}{2} = \$1.025$ in 6 months and $\left(1 + \frac{0.05}{2}\right)^2 = \$1.050625$ in 1 year, so the effective annual interest rate is $\sim 5.0625\%$.

In 6 months, \$1 grows to $(1+m)^6$, for some monthly rate m.

$$\begin{aligned} (1+m)^6 &= 1.025 \\ 1+m &= \sqrt[6]{1.025} \\ m &= \sqrt[6]{1.025} - 1 \end{aligned}$$

The effective monthly rate is $\sqrt[6]{1.025} - 1 \approx 0.206\%$.

Effective Interest Rates

Example

Calculate the effective annual interest rate and the equivalent weekly rate for an 12%/a interest rate, compounded semi-annually.

\$1 invested at 12%/a, compounded semi-annually, grows to $1 + \frac{0.12}{2} = \$1.06$ in 6 months and $\left(1 + \frac{0.12}{2}\right)^2 = \$1.1236$ in 1 year, so the effective annual interest rate is $\sim 12.36\%$.

In 26 weeks, \$1 grows to $(1+w)^{26}$, for some weekly rate w.

$$\begin{aligned} (1+w)^{26} &= 1.06 \\ 1+w &= \sqrt[26]{1.06} \\ w &= \sqrt[26]{1.06} - 1 \end{aligned}$$

The effective weekly rate is $\sqrt[26]{1.06} - 1 \approx 0.224\%$.

Mortgages

A *mortgage* is a type of loan offered from most banks and financial institutions. If a home-buyer cannot fully pay for the home, the balance due is mortgaged.

Mortgages are typically described using two criteria: an annual interest rate, i%/a, and an amortization term, n years.

By law, all fixed-rate mortgages in Canada are compounded semi-annually. Payments, however, are usually made weekly, bi-weekly or monthly, so there is a mismatch.

Typical amortization terms are 5, 10, 20 and 25 years, but other options are usually available.

A mortgage is simply a very large annuity, from which regular deductions are made. Thus, the value of a mortgage is represented by the present value of an annuity.

Mortgages

Example

A tenant in a highrise apartment pays \$1 200 per month. If he decides to buy and mortgage a house, paying the same amount each month for 25 years, what is the maximum mortgage he can afford if the current rate is 2.8%/a?

First, calculate the effective monthly rate using a semi-annual rate of 1.4%.

$$\begin{aligned} (1+m)^6 &= 1.014 \\ 1+m &= \sqrt[6]{1.014} \\ m &= \sqrt[6]{1.014} - 1 \end{aligned}$$

Therefore, $m \approx 0.002\,319\,838$, so the monthly rate is around 0.23%. To ensure a greater accuracy, however, it is better to use as many decimals as possible, or the exact value itself.

Mortgages

There will be a total of $12 \times 25 = 300$ monthly payments made over the 25 year term of the mortgage.

The maximum mortgage amount is the present value.

$$\begin{aligned} P &= 1\,200 \cdot \frac{1 - (1 + 0.002\,319\,838)^{-300}}{0.002\,319\,838} \\ &\approx \$259\,155.05 \end{aligned}$$

The tenant can afford a maximum mortgage of around \$260 000, given the conditions.

In reality, there are additional conditions (e.g. down payment, housing insurance, real estate fees, taxes, etc.) that must be met and paid for in order to secure a mortgage. These depend on many factors.

Mortgages

Example

What is the monthly payment for a \$180 000 mortgage, charging 4.6%/a interest for 25 years?

First, calculate the effective monthly rate using a semi-annual rate of 2.3%.

$$\begin{aligned} (1+m)^6 &= 1.023 \\ 1+m &= \sqrt[6]{1.023} \\ m &= \sqrt[6]{1.023} - 1 \end{aligned}$$

Therefore, $m \approx 0.003\,797\,105$, so the monthly rate is around 0.38%.

Mortgages

As in the previous example, a total of $12 \times 25 = 300$ payments will be made.

Use the formula for the present value of an annuity, with the regular payment R isolated.

$$\begin{aligned} R &= \frac{180\,000 \cdot 0.003\,797\,105}{1-(1+0.003\,797\,105)^{-300}} \\ &\approx \$1\,006.28 \end{aligned}$$

The monthly payment is approximately $1 006.28.

Mortgages

Example

How much interest is paid over 25 years on the $180 000 mortgage from the previous example?

There are 300 payments of $1 006.28, so a total of $300 \times \$1\,006.28 \approx \$301\,884$ was paid.

This means that $121 884 was paid in interest. Over 40% of all money paid went toward paying the bank, rather than paying off the mortgage.

Obviously, borrowing such a large amount of money for a considerably long amount of time is very costly.

Mortgages

Example

How would things change if the $180 000 mortgage is amortized over 20 years instead of 25?

This time, there are $20 \times 12 = 240$ payments made.

$$\begin{aligned} R &= \frac{180\,000 \cdot 0.003\,797\,105}{1-(1+0.003\,797\,105)^{-240}} \\ &\approx \$1\,144.27 \end{aligned}$$

The monthly payment is approximately $1 144.27, about $138 more each month.

However, the total amount paid over 20 years is $240 \times \$1\,144.27 \approx \$274\,624.80$. Paying off the mortgage sooner saves $27 259.20.

Mortgages

Example

How would things change if the $180 000 mortgage is amortized over 25 years, but paid bi-weekly?

This time, there are $25 \times 26 = 650$ payments made over the 25 life of the mortgage.

First, calculate the effective bi-weekly rate, given that there will be 13 payments made in 6 months.

$$\begin{aligned} (1+b)^{13} &= 1.023 \\ 1+b &= \sqrt[13]{1.023} \\ b &= \sqrt[13]{1.023}-1 \\ &\approx 0.001\,750\,722 \end{aligned}$$

Mortgages

Now we can calculate the bi-weekly payment.

$$\begin{aligned} R &= \frac{180\,000 \cdot 0.001\,750\,722}{1-(1+0.001\,750\,722)^{-650}} \\ &\approx \$463.96 \end{aligned}$$

After 25 years, a total of $650 \times \$463.96 \approx \$301\,574$ will have been paid. This is a savings of $310.

In this case, while the increased payment frequency has some effect on the total amount paid, the effect is minimal due to the overall size of the mortgage and the long amortization period.

Questions?