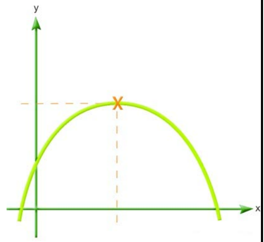
MCR3U: Functions

Minimums/Maximums of Quadratic Functions

Part 1: Completing the Square and Factoring

J. Garvin



Minimums/Maximums of Quadratics

Recall that a *quadratic* function has a *standard form* of $f(x) = ax^2 + bx + c$ where a, b and c are real numbers.

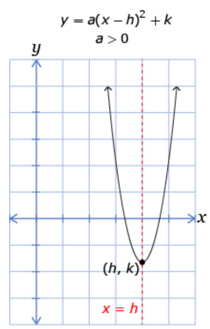
All quadratic functions have a domain of $\{x \in R\}$.

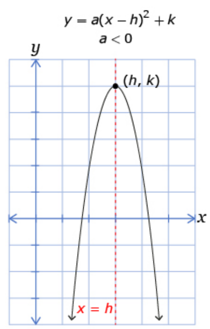
The range of a quadratic function, however, is restricted by the location of the vertex of the resulting *parabola*.

A quadratic function with vertex $V(h, k)$ has a *minimum* value of k when $a > 0$, and a range of $\{f(x) \in R \mid f(x) \geq k\}$.

Similarly, when $a < 0$ the *maximum* value is k and the range is $\{f(x) \in R \mid f(x) \leq k\}$.

Minimums/Maximums of Quadratics





Minimums/Maximums of Quadratics

The minimum/maximum or the range are easy to determine if we know the location of the vertex, but this is not readily apparent if the quadratic is in standard form.

A more useful representation is *vertex form*, $f(x) = a(x - h)^2 + k$, which specifies the location of the vertex, $V(h, k)$.

There are several methods available to convert a quadratic from standard form to vertex form.

Perfect Square Trinomials

For what value of c will $x^2 + bx + c$ be a perfect square?

Recall that a perfect square trinomial of the form $x^2 + bx + c$ satisfies the relationship $b = 2\sqrt{c}$.

Therefore, if $c = \left(\frac{b}{2}\right)^2$, then the trinomial is a perfect square.

Perfect Square Trinomials

Example

Determine the value of k such that $x^2 + 18x + k$ is a perfect square.

$$\begin{aligned} k &= \left(\frac{18}{2}\right)^2 \\ &= 9^2 \\ &= 81 \end{aligned}$$

Perfect Square Trinomials

Example

Determine the value of k such that $x^2 + \frac{4}{3}x + k$ is a perfect square.

$$\begin{aligned} k &= \left(\frac{\frac{4}{3}}{2}\right)^2 \\ &= \left(\frac{4}{6}\right)^2 \\ &= \left(\frac{2}{3}\right)^2 \\ &= \frac{4}{9} \end{aligned}$$

Completing the Square

One method of converting a quadratic function to vertex form is by a process called *Completing the Square* (CTS).

To do so, we factor out the leading coefficient from the terms of the trinomial involving the independent variable, then express what remains as a perfect square.

This is done by adding, and subtracting, a constant term of $\left(\frac{b}{2}\right)^2$. The subtraction ensures that the function is equivalent to the original.

Completing the Square

Example

Determine the minimum/maximum value of $f(x) = 2x^2 - 20x + 53$ by completing the square.

First, factor 2 out of the first two terms of the trinomial.

$$f(x) = 2(x^2 - 10x) + 53$$

Next, add and subtract a constant of $\left(\frac{-10}{2}\right)^2 = 25$ inside of the brackets.

$$f(x) = 2(x^2 - 10x + 25 - 25) + 53$$

Completing the Square

The first three terms inside of the brackets, $x^2 - 10x + 25$, now make a perfect square that factors as $(x - 5)^2$.

$$f(x) = 2([x - 5]^2 - 25) + 53$$

Expand and simplify.

$$\begin{aligned} f(x) &= 2(x - 5)^2 - 50 + 53 \\ &= 2(x - 5)^2 + 3 \end{aligned}$$

Therefore, the vertex is a at $(5, 3)$.

Since $a > 0$, the parabola has a minimum value.

The minimum value of $f(x)$ is 3, and occurs when $x = 5$.

Completing the Square

Your Turn

Determine the minimum/maximum value of $f(x) = -3x^2 + 21x - 35$ by completing the square.

$$\begin{aligned} f(x) &= -3(x^2 - 7x) - 35 \\ &= -3\left(x^2 - 7x + \left(\tfrac{-7}{2}\right)^2 - \left(\tfrac{-7}{2}\right)^2\right) - 35 \\ &= -3\left(\left[x - \tfrac{7}{2}\right]^2 - \left(\tfrac{49}{4}\right)\right) - 35 \\ &= -3\left(x - \tfrac{7}{2}\right)^2 + \tfrac{147}{4} - \tfrac{140}{4} \\ &= -3\left(x - \tfrac{7}{2}\right)^2 + \tfrac{7}{4} \end{aligned}$$

A maximum of $\frac{7}{4}$ occurs when $x = \frac{7}{2}$.

Partial Factoring

Another method of determining the location of the vertex is *partial factoring*.

Note that for any function $y = f(x)$, the graph of $y = f(x) + k$ would be the same, but with all points, including the vertex, moved upward (or downward) by a constant k.

The x-coordinate of the vertex would remain unchanged, so any quadratic function $f(x) = ax^2 + bx + c$ would have the same x-coordinate for its vertex as that of $g(x) = ax^2 + bx$.

By common factoring $g(x)$, we can determine the x-coordinate of the vertex, and use that value of x to find the associated y-coordinate.

Partial Factoring

Example

Determine the minimum/maximum value of $f(x) = 5x^2 + 20x + 23$ by partial factoring.

Let $g(x) = 5x^2 + 20x$. Then $g(x) = 5x(x + 4)$.

Set $g(x) = 0$ to find the zeroes (x-intercepts) of $g(x)$.

$$5x = 0 \qquad x + 4 = 0$$
$$x = 0 \qquad x = -4$$

Partial Factoring

Due to symmetry, the x-coordinate of the vertex is the average of these two values, or $\frac{0 + (-4)}{2} = -2$.

Therefore, we find $f(-2)$ to determine the location of the vertex.

$$\begin{aligned} f(-2) &= 5(-2)^2 + 20(-2) + 23 \\ &= 20 - 40 + 23 \\ &= 3 \end{aligned}$$

So the vertex is located at $(-2, 3)$, giving a minimum value of 3 when $x = -2$.

If we wanted to, we could write $f(x)$ in vertex form as $f(x) = 5(x + 2)^2 + 3$.

Partial Factoring

Your Turn

Determine the maximum/minimum value of $f(x) = 2x^2 - 24x + 59$.

$$\begin{aligned} g(x) &= 2x^2 - 24x \\ 0 &= 2x(x - 12) \\ x &= 0 \text{ or } 12 \\ \tfrac{0+12}{2} &= 6 \\ f(6) &= 2(6)^2 - 24(6) + 59 \\ &= -13 \end{aligned}$$

The minimum value of -13 occurs when $x = 6$.

Questions?