MCR3U: Functions

Linear/Quadratic Systems

Part 1: Solving Systems, Tangents

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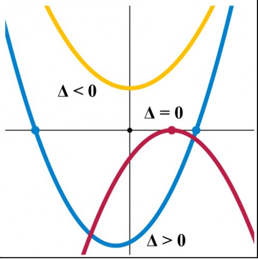


Linear/Quadratic Systems

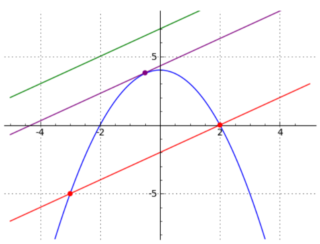
A *linear/quadratic system* is a system of two equations in which one is linear ($f(x) = mx + b$) and the other is quadratic ($f(x) = ax^2 + bx + c$).

Graphically, when solving a linear/quadratic system, we are interested in any points of intersection between the line and the parabola.

There are three cases that may occur.

1. The line may intersect the parabola in two locations (forming a *secant* between the two points)
2. The line may intersect the parabola at exactly one point (forming a *tangent* line at that point)
3. The line may miss the parabola completely

Linear/Quadratic Systems



Linear/Quadratic Systems

To solve a linear/quadratic system, we can use similar techniques for solving linear systems.

Consider the following linear/quadratic system.

$$f(x) = ax + b$$
$$g(x) = cx^2 + dx + e$$

Using substitution, $ax + b = cx^2 + dx + e$.

Rearranging and collecting like terms, this becomes $cx^2 + (d - a)x + (e - b) = 0$.

This is just a quadratic equation that can be solved using the usual techniques.

Linear/Quadratic Systems

Example

Determine any points of intersection of $f(x) = 2x - 4$ and $g(x) = x^2 - 5x + 2$.

Set the two equations equal to each other, then solve for x.

$$\begin{aligned} 2x - 4 &= x^2 - 5x + 2 \\ x^2 - 7x + 6 &= 0 \\ (x - 1)(x - 6) &= 0 \end{aligned}$$

Therefore, the line intersects the parabola when $x = 1$ and again when $x = 6$.

Linear/Quadratic Systems

To find the points of intersection, substitute $x = 1$ and $x = 6$ into the linear function.

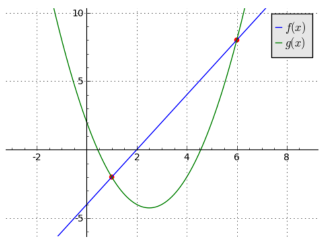
$$\begin{aligned} f(1) &= 2(1) - 4 \\ &= -2 \end{aligned} \qquad \begin{aligned} f(6) &= 2(6) - 4 \\ &= 8 \end{aligned}$$

The two points of intersection are $(1, -2)$ and $(6, 8)$.

The two values could have been substituted into the quadratic function as well, but using the linear function is generally easier.

Linear/Quadratic Systems

The graphs of $f(x) = 2x - 4$ and $g(x) = x^2 - 5x + 2$ confirm the two intersection points.



Linear/Quadratic Systems

Example

Determine any points of intersection of $f(x) = -x + 3$ and $g(x) = x^2 - 3x + 1$.

Set the two equations equal to each other as before.

$$-x + 3 = x^2 - 3x + 1$$
$$x^2 - 2x - 2 = 0$$

Since this quadratic does not factor, use the quadratic formula to solve for x.

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$
$$= 1 \pm \sqrt{3}$$

Linear/Quadratic Systems

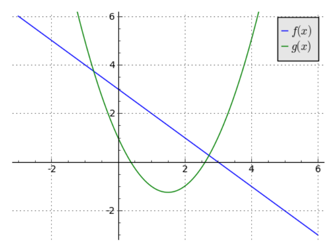
To find the points of intersection, substitute $x = 1 + \sqrt{3}$ and $x = 1 - \sqrt{3}$ into the linear function.

$$f(1 + \sqrt{3}) = -(1 + \sqrt{3}) + 3 = 2 - \sqrt{3} \quad f(1 - \sqrt{3}) = -(1 - \sqrt{3}) + 3 = 2 + \sqrt{3}$$

The two points of intersection are $(1 + \sqrt{3}, 2 - \sqrt{3})$ and $(1 - \sqrt{3}, 2 + \sqrt{3})$, or approximately $(2.7, 0.3)$ and $(-0.7, 3.7)$.

Linear/Quadratic Systems

The graphs of $f(x) = -x + 3$ and $g(x) = x^2 - 3x + 1$ confirm the two intersection points.



Linear/Quadratic Systems

Example

Determine any points of intersection of $f(x) = x - 7$ and $g(x) = x^2 - 4$.

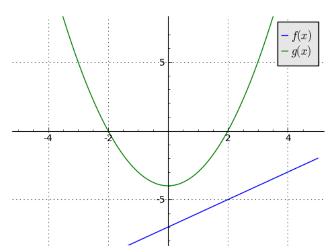
Set the two equations equal to each other, then solve for x.

$$x - 7 = x^2 - 4$$
$$x^2 - x + 3 = 0$$

Since the discriminant $(-1)^2 - 4(1)(3) = -11$ is negative, there are no real solutions to this system.

Linear/Quadratic Systems

The graphs of $f(x) = x - 7$ and $g(x) = x^2 - 4$ confirm that the two functions do not intersect.



Tangents

Example

Determine the point of tangency to $f(x) = -2(x+3)^2 + 5$ if the tangent has a slope of -4.

The tangent line has equation $g(x) = -4x + k$, for some value of k.

The quadratic is expressed in standard form, and must be expanded into standard form before substitution can be used.

$$\begin{aligned} f(x) &= -2(x+3)^2 + 5 \\ &= -2(x^2 + 6x + 9) + 5 \\ &= -2x^2 - 12x - 13 \end{aligned}$$

Tangents

Use substitution to solve the system.

$$\begin{aligned} -2x^2 - 12x - 13 &= -4x + k \\ 2x^2 + 8x + 13 + k &= 0 \end{aligned}$$

Since the tangent intersects the parabola exactly once, the solution to the system must be one real, repeated root.

Therefore, the discriminant must be equal to zero.

Using $a = 2$, $b = 8$ and $c = 13 + k$,

$$\begin{aligned} 8^2 - 4(2)(13+k) &= 0 \\ 64 - 104 - 8k &= 0 \\ -8k &= 40 \\ k &= -5 \end{aligned}$$

Tangents

The equation of the tangent is $g(x) = -4x - 5$.

The point of tangency occurs where $f(x)$ and $g(x)$ intersect.

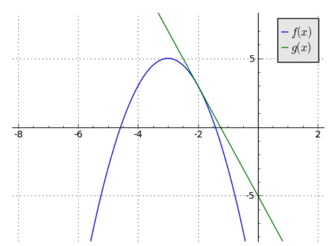
$$\begin{aligned} -2x^2 - 12x - 13 &= -4x - 5 \\ 2x^2 + 8x + 8 &= 0 \\ x^2 + 4x + 4 &= 0 \\ (x+2)^2 &= 0 \end{aligned}$$

This perfect square has one repeated, real root at $x = -2$.

When $x = -2$, $g(-2) = -4(-2) - 5 = 3$, so the point of tangency is $(-2, 3)$.

Tangents

The graphs of the parabola and its tangent are shown below.



Questions?