MCR3U: Functions

The Inverse of a Function

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The Inverse of a Function

A relation associates elements in the domain (independent variable) with those in the range (dependent variable).

When the dependent and independent variables (usually x and y) are swapped, the resulting relation is the *inverse* of the original relation.

Since functions are special cases of relations, the same definition applies.

Swapping the two variables means that the domain of a function becomes the range of its inverse, and the range of a function becomes the domain of its inverse.

For example, if a function has a domain of $\{x \in R \mid x > 4\}$ and a range of $\{f(x) \in R \mid x \leq 2\}$, then the inverse will have a domain of $\{x \in R \mid x \leq 2\}$ and a range of $\{f(x) \in R \mid x > 4\}$.

The Inverse of a Function

Example

Given the relation $R = \{(3,2),(-1,4),(5,0)\}$, Determine its inverse, I, and state the domain and range of I.

Swapping the values of x and y gives the inverse relation $I = \{(2,3),(4,-1),(0,5)\}$.

Since the domain and range of R are $D : \{3,-1,5\}$ and $R : \{2,4,0\}$, the domain and range of I are $D : \{2,4,0\}$ and $R : \{3,-1,5\}$.

The Inverse of a Function

Graphically, the inverse of a function is a reflection in the line $y = x$.

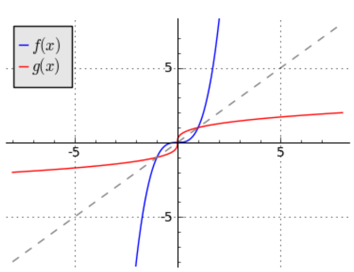
This makes it easy to graph a function's inverse by swapping the x- and y-coordinates for every point, and plotting the resulting new points.

This is often the best method to use when given a graph.

The Inverse of a Function

Example

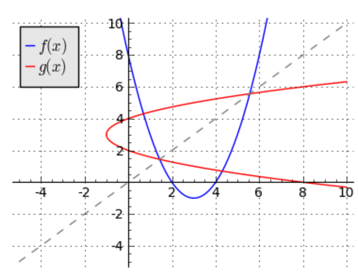
Graph the function $f(x) = x^3$ and its inverse.



The Inverse of a Function

Example

Graph the function $f(x) = (x-3)^2 - 1$ and its inverse.



The Inverse of a Function

In the last example, the inverse of the given function is not a function itself, since it fails the vertical line test.

The inverse of a function is *not* always a function.

One way to tell if the inverse of a function is a function as well is to use the Horizontal Line Test.

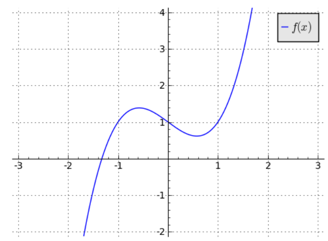
Horizontal Line Test

If it is possible to draw a horizontal line anywhere along a graph, such that the horizontal line intersects the graph more than once, then the inverse of the graph is not a function.

The Inverse of a Function

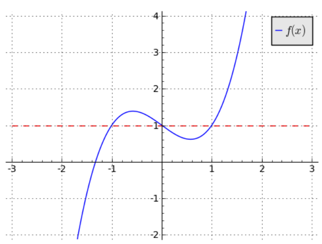
Example

Determine if the inverse of the function graphed below is also a function.



The Inverse of a Function

Since a horizontal line can be drawn on the graph that will intersect the function more than once, the inverse of the function is *not* a function itself.



The Inverse of a Function

It is possible to determine the equation of a function's inverse by swapping all instances of the independent and dependent variable.

Once swapped, isolate the new dependent variable.

We use the notation $f^{-1}(x)$ to denote the inverse of a function.

If the inverse is not a function itself, we typically do not use this notation.

The Inverse of a Function

Example

Determine the equation of the inverse of $f(x) = \dfrac{2}{x-3}$

$$\begin{aligned} x &= \tfrac{2}{y-3} \\ y - 3 &= \tfrac{2}{x} \\ y &= \tfrac{2}{x} + 3 \end{aligned}$$

The equation of the inverse is $f^{-1}(x) = \frac{2}{x} + 3$.

The Inverse of a Function

Example

Determine the equation of the inverse of $g(x) = 3(x-2)^2 - 4$

$$\begin{aligned} x &= 3(y-2)^2 - 4 \\ x + 4 &= 3(y-2)^2 \\ \tfrac{x+4}{3} &= (y-2)^2 \\ \pm\sqrt{\tfrac{x+4}{3}} &= y - 2 \\ 2 \pm \sqrt{\tfrac{x+4}{3}} &= y \end{aligned}$$

The equation of the inverse is $y = 2 \pm \sqrt{\frac{x+4}{3}}$.

The Inverse of a Function

Sometimes it is necessary to state restrictions on the domain of the inverse, such that it corresponds to the range of the original function.

For example, squaring and square-rooting are inverse operations. For this reason, when $f(x) = x^2$ is reflected in the line $y = x$, it looks very similar to $g(x) = \sqrt{x}$.

The one difference, however, is that $g(x) = \sqrt{x}$ is only one half of the graph of $f(x) = x^2$.

If we restrict the domain of $f(x) = x^2$ such that $x \leq 0$, then its inverse is $g(x) = \sqrt{x}$.

The Inverse of a Function

Example

Determine the equation of the inverse of $h(x) = -2\sqrt{x-5} + 1$, and state restrictions on its domain.

$$
\begin{aligned}
x &= -2\sqrt{y-5} + 1 \\
x - 1 &= -2\sqrt{y-5} \\
-\tfrac{x-1}{2} &= \sqrt{y-5} \\
\left(\tfrac{x-1}{2}\right)^2 &= y - 5 \\
\left(\tfrac{x-1}{2}\right)^2 + 5 &= y
\end{aligned}
$$

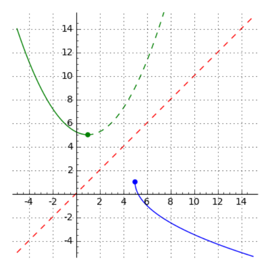
The Inverse of a Function

The domain and range of $h(x)$ are $\{x \in R \mid x \geq 5\}$ and $\{h(x) \in R \mid h(x) \leq 1\}$ respectively.

The range becomes the domain for the inverse.

Therefore, the equation of the inverse is $h^{-1}(x) = \left(\frac{x-1}{2}\right)^2 + 5$, with domain $\{x \in R \mid x \leq 1\}$ and range $\{h^{-1}(x) \in R \mid h^{-1}(x) \geq 5\}$.

The Inverse of a Function



The solid lines show the function (blue) and its inverse (green), while the dotted green line shows how the graph of $h^{-1}(x) = \left(\frac{x-1}{2}\right)^2 + 5$ would continue if the domain was not limited.

Questions?