MCR3U: Functions

Geometric Series

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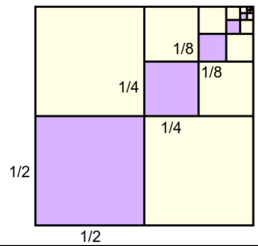

Geometric Series

Consider the series $2 + 6 + 18 + 54 + \ldots$.

Since there is no common difference between the terms, the series is not arithmetic.

There is, however, a common ratio between terms: $6/2 = 3$, $18/6 = 3$, etc.

This indicates that the relationship is geometric.

Geometric Series

If a sequence is geometric, then the sum of its terms is an geometric series.

Geometric Series

To derive a formula for the sum of the first n terms of a geometric series, begin by writing the series in terms of the common ratio r.

$$S = t_1 + t_1 \cdot r + t_1 \cdot r^2 + t_1 \cdot r^3 + \ldots + t_1 \cdot r^{n-1}$$

Create a second representation of the series by multiplying each term by r.

$$r \cdot S = t_1 \cdot r + t_1 \cdot r^2 + t_1 \cdot r^3 + \ldots + t_1 \cdot r^{n-1} + t_1 \cdot r^n$$

Subtracting the first representation from the second, matching-coloured terms cancel. Factor and solve for S.

$$\begin{aligned} r \cdot S - S &= t_1 \cdot r^n - t_1 \\ S(r-1) &= t_1(r^n - 1) \\ S &= t_1 \cdot \frac{r^n - 1}{r - 1} \end{aligned}$$

Geometric Series

Sum of n Terms in a Geometric Series

For a geometric series, the sum of the first n terms, S_n, is given by $S_n = t_1 \cdot \frac{r^n - 1}{r - 1}$ where t_1 is the initial term and r the common ratio.

Unlike the formula for an arithmetic series, we do not need to know the value of the nth term to calculate the sum.

Example

Determine the sum of the first 10 terms of the series $4 + 8 + 16 + \ldots$.

Use $t_1 = 4$, $r = 2$ and $n = 10$ with the formula above.

$$\begin{aligned} S_{10} &= 4 \cdot \frac{2^{10} - 1}{2 - 1} \\ &= 4\,092 \end{aligned}$$

Geometric Series

Remember that a geometric sequence represents exponential growth, and increases (or decreases) rapidly.

A geometric series will also grow (or shrink) very quickly.

Example

Determine the sum of the first 20 terms of the series $5 + 20 + 80 + \ldots$.

Use $t_1 = 5$, $r = 4$ and $n = 20$.

$$\begin{aligned} S_{20} &= 5 \cdot \frac{4^{20} - 1}{4 - 1} \\ &= 1\,832\,519\,379\,625 \end{aligned}$$

Geometric Series

If $r < 0$, the terms of a geometric sequence alternate signs.

The terms of a geometric series with alternating signs will cause the sum to increase, then decrease, then increase, etc.

Example

Determine the sum of the first 12 terms of the series $3 - 6 + 12 - 24 + \ldots$.

Use $t_1 = 3$, $r = -2$ and $n = 12$.

$$\begin{aligned} S_{12} &= 3 \cdot \frac{(-2)^{12} - 1}{-2 - 1} \\ &= -4\,095 \end{aligned}$$

Compare this to the sum of the first 12 terms of the sequence $3 + 6 + 12 + 24 + \ldots$, which is $12\,285$.

Geometric Series

Example

Determine the sum of the geometric series $1+3+\ldots+6\,561$.

First we need to determine the number of terms, n.

$$\begin{aligned} 6\,561 &= 1 \cdot 3^{n-1} \\ 3^8 &= 3^{n-1} \\ 8 &= n-1 \\ n &= 9 \end{aligned}$$

Now we can use the geometric series formula.

$$\begin{aligned} S_9 &= 1 \cdot \frac{3^9-1}{3-1} \\ &= 9\,841 \end{aligned}$$

Geometric Series

Example

In a geometric series, the third term is -50 and the sixth term is $6\,250$. Determine the sum of the first ten terms.

Use $t_3 = -50$ and $t_6 = 6\,250$ to solve for r.

$$\begin{aligned} -50 \cdot r^3 &= 6\,250 \\ r^3 &= -125 \\ r &= -5 \end{aligned}$$

Use the formula for the general term with $r=-5$, $n=3$ and $t_3=-50$.

$$\begin{aligned} -50 &= t_1(-5)^{3-1} \\ t_1 &= -2 \end{aligned}$$

Geometric Series

Finally, solve for S_{10}.

$$\begin{aligned} S_{10} &= -2 \cdot \frac{(-5)^{10}-1}{-5-1} \\ &= 3\,255\,208 \end{aligned}$$

While it is fairly easy to solve problems like this one involving multiple *terms*, it is usually more complicated to solve problems involving multiple *sums*, since these often lead to situations that require factoring higher-order polynomials.

Geometric Series

Example

In the final round of a game show, a contestant spins a wheel containing 20 equally-sized sectors. Half of these say "win" and half say "lose". If the contestant spins "win", his score is increased by an amount double the current score. If the contestant spins "lose", he loses all points. He may spin any number of times. If the contestant currently has 40 points, and spins "win" four times in a row, what will his score be?

Use $t_1 = 40$, $r = 2$ and $n = 5$ to find S_5.

$$\begin{aligned} S_5 &= 40 \cdot \frac{2^5-1}{2-1} \\ &= 1\,240 \end{aligned}$$

The contestant's score will be 1 240 points.

Questions?