MCR3U: Functions

Geometric Sequences

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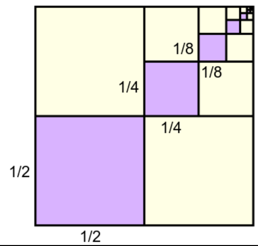

Arithmetic Sequences

Recap

Determine the value of the 250th term of the arithmetic sequence beginning $476, 453, 430, \ldots$.

$t_1 = 476$, $n = 250$ and $d = 453 - 476 = -23$.

$$\begin{aligned} t_{250} &= 476 + (250 - 1)(-23) \\ &= -5\,251 \end{aligned}$$

Therefore, $t_{250} = -5\,251$.

Geometric Sequences

Consider the sequence $2, 8, 32, 128, \ldots$.

The sequence is not arithmetic, since there is no common difference (e.g. $8 - 2 = 6$, but $32 - 8 = 24$).

There is, however, a pattern involving multiplication, since $2 \cdot 4 = 8$, $8 \cdot 4 = 32$, etc.

Geometric Sequences

If a sequence has a *common ratio* between any two successive terms, it is called a *geometric sequence*.

Geometric Sequences

Example

Determine whether the sequence beginning $2, 14, 98, 686, \ldots$ is geometric or not.

Since there is a common ratio of 7 between any two terms (e.g. $14/2 = 7$), the sequence is geometric.

Example

Determine whether the sequence beginning $1, -2, 4, -8, \ldots$ is geometric or not.

As before, there is a common ratio between any two terms (e.g. $-8/4 = -2$), so the sequence is geometric. Note that it is possible for the signs to alternate between terms in a geometric sequence.

Geometric Sequences

Like with arithmetic sequences, a geometric sequence can be extended using the common ratio.

Example

Determine the next three terms of the geometric sequence beginning $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots$.

The common ratio is $\frac{1}{6} \div \frac{1}{3} = \frac{1}{2}$. Therefore, the next three terms are $t_4 = \frac{1}{12} \cdot \frac{1}{2} = \frac{1}{24}$, $t_5 = \frac{1}{24} \cdot \frac{1}{2} = \frac{1}{48}$ and $t_6 = \frac{1}{48} \cdot \frac{1}{2} = \frac{1}{96}$.

A formula for the general term of a geometric sequence can also be used when we wish to determine values further from the initial value.

Geometric Sequences

Consider the sequence $t_1, t_2, t_3, \ldots, t_{n-1}, t_n$. As the sequence is geometric, there is a common ratio between terms.

Thus, $t_2 = t_1 \cdot r$, $t_3 = t_2 \cdot r = (t_1 \cdot r) \cdot r = t_1 \cdot r^2$, etc.

Therefore, the geometric sequence can be rewritten as $t_1, t_1 \cdot r, t_1 \cdot r^2, \ldots, t_1 \cdot r^{n-2}, t_1 \cdot r^{n-1}$.

General Term of a Geometric Sequence

The general term of a geometric sequence is given by $t_n = t_1 \cdot r^{n-1}$, where t_n is the value of the nth term, t_1 is the value of the 1st term, and r is the common ratio between successive terms.

Since a geometric sequence represents exponential growth, the formula for the general term resembles that for an exponential relation.

Geometric Sequences

Example

Develop a formula for the general term of the geometric sequence that begins $3, 12, 48, \ldots$, then determine the value of the 12th term.

$t_1 = 3$ and $r = 12/3 = 4$. Therefore, an equation is $t_n = 3 \cdot 4^{n-1}$.

Using $n = 12$, $t_{12} = 3 \cdot 4^{12-1} = 12\,582\,912$.

Like exponential functions, geometric sequences increase or decrease rapidly.

Geometric Sequences

Example

In a geometric sequence, the 5th term is 1 250 and the 7th term is 31 250. Determine the value of the 12th term.

To go from the 5th term to the 7th term, it is necessary to multiply by the common ratio twice.

$$1\,250 \cdot r^2 = 31\,250$$
$$r^2 = 25$$
$$r = \pm 5$$

In this case, there are two possible values for the common ratio. If $r = 5$, then even terms in the sequence will be positive. If $r = -5$, even terms will be negative. All odd terms, including t_1, will be positive.

Geometric Sequences

Solve for t_1 using $r = 5$.

$$1\,250 = t_1 \cdot 5^{5-1}$$
$$t_1 = 2$$

When $r = 5$, $t_{12} = 2 \cdot 5^{12-1} = 97\,656\,250$.

When $r = -5$, $t_{12} = 2\,(-5)^{12-1} = -97\,656\,250$.

There are two possible answers for this sequence. This is not always the case.

Geometric Sequences

Example

Determine the number of terms in the geometric sequence $\frac{3}{8}, \frac{3}{4}, \ldots, 12\,288$.

The common ratio is $\frac{3}{4} \div \frac{3}{8} = 2$.

Substitute $t_1 = \frac{3}{8}$, $r = 2$ and $t_n = 12\,288$ into the formula for the general term.

$$12\,288 = \tfrac{3}{8} \cdot 2^{n-1}$$
$$32\,768 = 2^{n-1}$$
$$2^{15} = 2^{n-1}$$
$$15 = n - 1$$
$$n = 16$$

Therefore, the sequence has 16 terms.

Geometric Sequences

Example

A game is played where a player spins a spinner, which is divided into sectors with various values. One of these sectors increases the player's score by a factor of ten. If the player has thirteen points, and miraculously lands on this sector four times in a row, how many points will (s)he have?

If the player's score is t_1, then after four spins, the player's score will be t_5.

Substitute $t_1 = 13$, $r = 10$ and $n = 5$ into the equation for the general term, and solve for t_5.

$$t_5 = 13 \cdot 10^{5-1}$$
$$= 130\,000$$

There player will have 130 000 points after the five spins.

Geometric Sequences