MCR3U: Functions

Exponent Laws

Integral Exponents

J. Garvin

Exponent Laws

In previous courses, we have explored the three "fundamental" exponent laws:

- the Product Rule,
- the Quotient Rule, and
- the Power of a Power Rule.

You may have also been introduced to negative or zero exponents.

In this unit, we will explore some additional exponent laws related to rational exponents.

Before we do that, however, we must review the five rules listed above.

Multiplying Powers with Like Bases

Product Rule of Exponents

When two powers with the same base are multiplied, add the exponents.

$$x^a \cdot x^b = x^{a+b}$$

Consider the powers x^a and x^b.

$$\begin{aligned} x^a \cdot x^b &= \underbrace{x \cdot x \cdot \ldots \cdot x}_{a \text{ times}} \cdot \underbrace{x \cdot x \cdot \ldots \cdot x}_{b \text{ times}} \\ &= x^{a+b} \end{aligned}$$

Dividing Powers with Like Bases

Quotient Rule of Exponents

When two powers with the same base are divided, subtract the exponents.

$$\frac{x^a}{x^b} = x^{a-b}$$

Consider the powers x^a and x^b.

$$\begin{aligned} \frac{x^a}{x^b} &= \frac{\overbrace{x \cdot x \cdot x \cdot \ldots \cdot x}^{a \text{ times}}}{\underbrace{x \cdot x \cdot \ldots \cdot x}_{b \text{ times}}} \\ &= x^{a-b} \end{aligned}$$

So far, $a > b$, but this will change shortly.

Multiplying/Dividing Powers

Example

Simplify $a^5 \cdot a^9$.

$$\begin{aligned} a^5 \cdot a^9 &= a^{5+9} \\ &= a^{14} \end{aligned}$$

Example

Simplify $\frac{b^8}{b^6}$.

$$\begin{aligned} \frac{b^8}{b^6} &= b^{8-6} \\ &= b^2 \end{aligned}$$

Raising Powers to an Exponent

Power of a Power Rule

When one power is raised to another exponent, multiply the exponents.

$$(x^a)^b = x^{ab}$$

Consider the powers x^a and x^b.

$$\begin{aligned} (x^a)^b &= \underbrace{x^a \cdot x^a \cdot \ldots \cdot x^a}_{b \text{ times}} \\ &= x^{ab} \end{aligned}$$

Raising Powers to an Exponent

Example

Simplify $\left(p^2q^3\right)^8$.

$$\begin{aligned}\left(p^2q^3\right)^8 &= p^{2\cdot 8}q^{3\cdot 8}\\ &= p^{16}q^{24}\end{aligned}$$

Raising Powers to an Exponent

Example

Simplify $\left(\frac{a^2b^4}{c^3}\right)^5$.

$$\begin{aligned}\left(\frac{a^2b^4}{c^3}\right)^5 &= \frac{a^{2\cdot 5}b^{4\cdot 5}}{c^{3\cdot 5}}\\ &= \frac{a^{10}b^{20}}{c^{15}}\end{aligned}$$

Zero Exponents

Zero Exponent Rule

Any non-zero base with a zero exponent has a value of 1.

$$x^0 = 1, x \neq 0$$

This can be shown using the quotient rule described earlier.

Consider the expression $\frac{x^n}{x^n}$. Using the quotient rule, $\frac{x^n}{x^n} = x^{n-n} = x^0$.

At the same time, $\frac{x^n}{x^n}$ indicates that x^n is divided by itself, which is clearly 1.

Zero Exponents

What about 0^0?

Using the quotient rule, $0^0 = \frac{0^n}{0^n}$.

Since this expression involves division by zero, 0^0 is not defined.

Zero Exponents

Example

Simplify $(3-7)^0$.

Since the expression inside of the brackets is not zero, $(3-7)^0 = 1$.

Example

Simplify $(2x-1)^0$.

If $x = \frac{1}{2}$, then the expression inside of the brackets will be zero, resulting in 0^0. For all other values of x, this problem does not arise.

We can say $(2x-1)^0 = 1, x \neq \frac{1}{2}$. We have seen *restrictions* on the variable before, when we covered rational expressions.

Negative Exponents

Negative Exponent Rule

A power with a negative exponent is equivalent to the reciprocal of that power with a positive exponent.

$$x^{-n} = \frac{1}{x^n}$$

We can demonstrate this to be true by using zero exponents.

$$\begin{aligned}x^{-n} &= x^{0-n}\\ &= \frac{x^0}{x^n}\\ &= \frac{1}{x^n}\end{aligned}$$

Negative Exponents

Example

Express a^{-5} using a positive exponent.

$$a^{-5} = \frac{1}{a^5}$$

Example

Express $\left(\frac{ab}{c}\right)^{-3}$ using positive exponents.

$$\left(\frac{ab}{c}\right) = \frac{1}{\left(\frac{ab}{c}\right)^3} = \frac{c^3}{a^3b^3}$$

Combining Exponent Laws

The previous exponent laws can be combined to simplify expressions to a single power.

Whenever possible, look for shortcuts like reducing or combining powers first.

Most of the time, it is preferable to express answers using positive exponents.

Combining Exponent Laws

Example

Simplify $\left(x^3 \cdot x^4\right)^2$.

$$\left(x^3 \cdot x^4\right)^2 = \left(x^{3+4}\right)^2 = \left(x^7\right)^2 = x^{7\cdot 2} = x^{14}$$

Combining Exponent Laws

Example

Simplify $\left(\frac{n^2 \cdot n^5}{n^4}\right)^2 \left(\frac{n^8}{n^6}\right)$.

$$\left(\frac{n^2 \cdot n^5}{n^4}\right)^2 \left(\frac{n^8}{n^6}\right) = \left(\frac{n^7}{n^4}\right)^2 \left(\frac{n^8}{n^6}\right) = \left(n^3\right)^2 \left(n^2\right) = n^6 \cdot n^2 = n^8$$

Combining Exponent Laws

Example

Simplify $\frac{\left(p^3q^{-2}\right)^4}{p^2q^{-3}}$ using positive exponents only.

$$\frac{\left(p^3q^{-2}\right)^4}{p^2q^{-3}} = \frac{p^{12}q^{-8}}{p^2q^{-3}} = p^{10}q^{-5} = \frac{p^{10}}{q^5}$$

Questions?