MCR3U: Functions

# Compound Interest

## Part 1: Future Value

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## Compound Interest

In the past, many investments used *simple interest*, in which investments earn a fixed amount of interest, based on the principal, at the end of each year.

Today, nearly all investments use some variant of *compound interest*, in which interest is earned on previous interest.

Consider \$100, invested at 5%/a interest (/a is notation for "compounded annually") for 3 years.

At the end of the first year, the investment will earn \$5 in interest, for a total value of \$105.

At the end of the second year, the investment will earn 5% on the \$105, or \$5.25, bringing the total to \$110.25.

At the end of the third year, the investments earns 5% on the \$110.25, or \$5.51, resulting in a final balance of \$115.76.





## Compound Interest

If some amount $P$ is invested into an account paying $i$%/a interest for $n$ years, then the investment will have a *future value* of:

- $FV = P(1+i)$ after the first year,
- $FV = P(1+i)(1+i) = P(1+i)^2$ after the second year,
- $FV = P(1+i)^2(1+i) = P(1+i)^3$ after the third year,
- ...
- $FV = P(1+i)^{n-1}(1+i) = P(1+i)^n$ after the $n$th year.

**Future Value of an Investment, Compounded Annually**

If some principal amount, $P$, is invested at $i$%/a for $n$ years, then its future value, $FV$, is given by $FV = P(1+i)^n$.





## Compound Interest

**Example**

\$6 200 is invested into an account that pays 2.8%/a. How much will the investment be worth after 5 years?

Use $P = \$6\,200$, $i = 0.028$ and $n = 5$.

$$\begin{aligned} FV &= 6\,200\,(1+0.028)^5 \\ &\approx \$7\,117.99 \end{aligned}$$

Using simple interest, the investment would earn $\$6\,200 \times 0.028 \times 5 = \$868$ in interest, for a total value of \$7 068 at the end of 5 years.

Compound interest results in (marginally) more interest than simple interest for this investment.





## Compound Interest

**Example**

What annual interest rate is required to double the value of an investment, compounded annually, in 20 years?

Assume we invest \$1 into account, such that it grows to \$2 in 20 years.

$$\begin{aligned} 2 &= 1\,(1+i)^{20} \\ \sqrt[20]{2} &= 1+i \\ i &= \sqrt[20]{2} - 1 \end{aligned}$$

Since $\sqrt[20]{2} - 1 \approx 0.035$, an annual interest rate of just over 3.5% would be necessary to double the investment in the specified time.





## Compound Interest

**Example**

How much time is required to earn \$20 000 interest on a \$50 000 investment, in an account paying $4\frac{3}{4}$%/a interest?

If \$20 000 is paid in interest, then the account must be worth $\$50\,000 + \$20\,000 = \$70,000$ at the end of the specified time.

Use $P = \$50\,000$, $FV = \$70,000$, and $i = 0.0475$.

$$\begin{aligned} 70\,000 &= 50\,000\,(1+0.0475)^t \\ 1.4 &\approx 1.0475^t \end{aligned}$$

At this stage, use "guess-and-check" to determine an appropriate value for $t$.

If $t = 7.25$, then the account will be worth $50\,000\,(1+0.0475)^{7.25} \approx \$69\,998.18$.

## Compound Interest

If an investment has a different *compounding period*, then two changes must be made to the formula.

- the time is multiplied by the number of compounding periods, and
- the interest rate is divided by the frequency of the compounding.

For example, if an investment is made at 6%/a for two years, but is compounded monthly instead of annually, then there are $2 \times 12 = 24$ compounding periods, each with an *effective interest rate* of $6 \div 12 = 0.5$%/a.

**Future Value of an Investment, Compounded Regularly**

If some principal amount, $P$, is invested at $i$%/a for $n$ years, then its future value, $FV$, is given by $FV = P\left(1+\frac{i}{c}\right)^{nc}$, where $c$ is the number of compounding periods per year.





## Compound Interest

**Example**

An investment of \$5 000 is made at 3%/a, compounded monthly. How much is it worth after 10 years?

Use $P = 5\,000$, $i = 0.03$, $n = 10$ and $c = 12$.

$$\begin{aligned} FV &= 5\,000\left(1+\tfrac{0.03}{12}\right)^{10\times12} \\ &\approx \$6\,746.77 \end{aligned}$$

**Example**

How much interest was earned on the investment?

The interest is the amount added to the principle, so the investment earned $6\,746.77 - 5\,000 \approx \$1\,746.77$ in interest.





## Compound Interest

**Example**

Which investment will be worth more after 20 years:

- \$10 000 at 4.5%/a, compounded monthly, or
- \$8 000 at 6%/a, compounded weekly?

The first investment is worth
$FV = 10\,000\left(1+\frac{0.045}{12}\right)^{20\times12} \approx \$24\,554.66$.

The second investment is worth
$FV = 8\,000\left(1+\frac{0.06}{52}\right)^{20\times52} \approx \$26\,542.57$.

The second investment will be worth \$1 987.91 more than the first after the same amount of time, due to the increased compounding frequency and larger interest rate.





## Compound Interest

**Example**

A mutual fund is being discontinued, and all investments into that fund must choose between one of two options:

- a $50-50$ split between two accounts, one paying 3.6%/a, compounded monthly, and the other paying 2.1%/a, compounded weekly, or
- a full investment into an account paying 2.9%/a, compounded bi-weekly.

If both options have 5-year terms, which is "better" for an investment currently valued at \$12 000?





## Compound Interest

For the first option, the \$12 000 is split into two sums of \$6 000 each, and invested into the two accounts.

$$\begin{aligned} FV_1 &= 6\,000\left(1+\tfrac{0.036}{12}\right)^{5\times12} \\ &\approx \$7\,181.37 \end{aligned} \qquad \begin{aligned} FV_2 &= 6\,000\left(1+\tfrac{0.021}{52}\right)^{5\times52} \\ &\approx \$6\,664.12 \end{aligned}$$

The total value is $\$7\,181.37 + \$6\,664.12 \approx \$13\,845.49$.

The second option invests the \$12 000 into a single fund.

$$\begin{aligned} FV &= 12\,000\left(1+\tfrac{0.029}{26}\right)^{5\times26} \\ &\approx \$13\,871.35 \end{aligned}$$

The second investment is the "better" option, in that it will earn more interest in 5 years.





## Questions?