MCR3U: Functions

Applications of Exponential Functions

Part 1: Simple Exponential Growth/Decay

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Applications of Exponential Functions

Many scenarios can be modelled using exponential functions, including:

- population increase/decrease,
- radioactive decay,
- appreciation/depreciation of monetary values, etc.

In most cases, these functions will have the form $f(t) = a \cdot b^{t/p}$, where:

- $f(t)$ is the final value,
- a is the initial value,
- b is the base (e.g. doubling, halving),
- t is the time elapsed, and
- p is the period (the time taken for the action specified by the base to occur)

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Example

Alfred has an ant colony, whose population doubles every 5 months. If he begins with 100 ants, approximately how many ants should there be after 1 year?

Assuming no ants are killed or eaten (by ants or by Alfred), we can use the formula $P(t) = 100 \cdot 2^{t/5}$, where P is the population and t represents the time in months.

After 1 year, or 12 months, the population should be $P(12) = 100 \cdot 2^{12/5} \approx 528$ ants.

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Example

How long will it take until the same colony reaches a population of 3 200 ants?

We want to determine the time, given a final population. Substitute $P(t) = 3\,200$ into the equation and solve for t.

$$\begin{aligned} 3\,200 &= 100 \cdot 2^{t/5} \\ 32 &= 2^{t/5} \\ 2^5 &= 2^{t/5} \\ 5 &= t/5 \\ t &= 25 \end{aligned}$$

It will take 25 months, or just over 2 years for the population to reach 3 200 ants.

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Example

How long will it take until the same colony reaches a population of 10 000 ants?

Like before, substitute $P(t) = 10\,000$ into the equation.

$$\begin{aligned} 10\,000 &= 100 \cdot 2^{t/5} \\ 100 &= 2^{t/5} \end{aligned}$$

Since 100 is not a power of 2, we can't use a change of base like the previous question. Instead, use "guess and check".

Since $2^6 = 64$ and $2^7 = 128$, t should be somewhere between 30 and 35.

When $t = 33.22$, then $P(33.22) \approx 100$. Therefore, it will take slightly longer than 33 months, or $2\frac{3}{4}$ years.

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Example

Radioactive substances lose mass as they emit particles into the air. The *half-life* of a substance is the time it takes for that substance to decrease by half of its original mass. If 30 grams of a substance has a half-life of 15 years, how much will remain after 50 years?

Use the equation $M(t) = 30\left(\frac{1}{2}\right)^{t/15}$, where M is the mass and t is the time in years.

After 50 years, the mass will be $M(50) = 30\left(\frac{1}{2}\right)^{50/15} \approx 2.98$ grams.

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Example

How much of the radioactive substance from the previous example was present 10 years ago?

Since we want to know a mass in the past, we can use a negative exponent to move "backward" 10 years in time.

Therefore, 10 years ago the mass would have been $M(t) = 30\left(\frac{1}{2}\right)^{-10/15} \approx 47.62$ grams.

It is also possible to reverse the process, and treat the scenario as if the substance is doubling rather than halving.

In this case, using the function $M(t) = 30 \cdot 2^{t/15}$ where $t = 10$, we obtain $M(10) = 30 \cdot 2^{10/15} \approx 47.62$ grams.

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A *third* method is to treat the 30 grams as a final value, and solve for the initial value.

$$\begin{aligned} 30 &= a\left(\tfrac{1}{2}\right)^{10/15} \\ a &= \frac{30}{\left(\frac{1}{2}\right)^{2/3}} \\ a &\approx 47.62 \end{aligned}$$

All three methods are equivalent. Use whichever one is most comfortable.

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Example

A scientist discovers a new strain of bacteria. She isolates 50 bacteria in a petri dish. After 24 hours, the population has grown to 800 bacteria. What is the doubling period?

In this case we want to know the value of p, not t.

$$\begin{aligned} 800 &= 50 \cdot 2^{24/p} \\ 16 &= 2^{24/p} \\ 2^4 &= 2^{24/p} \\ 4 &= \tfrac{24}{p} \\ p &= 6 \end{aligned}$$

The bacterial population doubles every 6 hours.

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Example

The value of a painting increases exponentially with time. In 1974, it sold for \$3 000. In 2014 it sold for \$165, 000. By how much does the painting increase its value each year?

We know the initial and final values of the painting (\$3 000 and \$165 000), and also the time elapsed (40 years). We need to determine the base, b.

$$\begin{aligned} 165\,000 &= 3\,000 \cdot b^{40} \\ 55 &= b^{40} \\ b &= \sqrt[40]{55} \\ b &\approx 1.1054 \end{aligned}$$

The painting has a value of approximately 110.54% at the end of each year, or an increase of approximately 10.54%.

Questions?