MCR3U: Functions

Factoring Review (Part 2)

Complex Trinomials, Perfect Squares and Differences of Squares

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Complex Trinomials

A trinomial with a leading coefficient other than 1 is a *complex trinomial*.

Since the leading coefficient is not 1, at least one of the two binomials that produced it must have a leading coefficient that is not 1.

$$\begin{aligned}(2x-3)(x-5) &= 2x\cdot x - 5\cdot 2x - 3\cdot x - 3\cdot(-5)\\ &= 2x^2 - 10x - 3x + 15\\ &= 2x^2 - 13x + 15\end{aligned}$$

Factoring Complex Trinomials

Take a closer look at the second line of the previous example.

$$2x^2 - 10x - 3x + 15$$

The two central values, -10 and -3, have a product of 30, which is the product of the first and last terms of the trinomial (2×15).

They have a sum of -13, which is the central term of the trinomial.

We can use this knowledge in a technique called *decomposition* to convert a complex trinomial to a form that can be factored by grouping.

Factoring Complex Trinomials

Example

Factor $3x^2 + 13x + 4$.

We break down the central term, $13x$, into two terms that have coefficients with a product of $3\times 4 = 12$ and sum of 13.

$$\begin{aligned}3x^2 + 13x + 4 &= 3x^2 + x + 12x + 4\\ &= x(3x+1) + 4(3x+1)\\ &= (3x+1)(x+4)\end{aligned}$$

Factoring Complex Trinomials

Your Turn

Factor $6x^2 + x - 15$.

We break down the central term, x, into two terms that have coefficients with a product of $6\times(-15) = -90$ and sum of 1.

$$\begin{aligned}6x^2 + x - 15 &= 6x^2 + 10x - 9x - 15\\ &= 2x(3x+5) - 3(3x+5)\\ &= (3x+5)(2x-3)\end{aligned}$$

Factoring Complex Trinomials

Your Turn

Factor $16x^2 - 24x + 9$.

We break down the central term, $-24x$, into two terms that have coefficients with a product of $16\times 9 = 144$ and sum of -24.

$$\begin{aligned}16x^2 - 24x + 9 &= 16x^2 - 12x - 12x + 9\\ &= 4x(4x-3) - 3(4x-3)\\ &= (4x-3)(4x-3)\\ &= (4x-3)^2\end{aligned}$$

Perfect Squares Trinomials

The last example is called a *perfect square trinomial*, since it can be factored as the square of a binomial.

While perfect squares can be factored via decomposition, there is a shortcut that we can use by noting a relationship between the coefficients.

Factoring a Perfect Square Trinomial

Given a trinomial $ax^2 + bx + c$, if $|b| = 2\sqrt{a}\sqrt{c}$ then the trinomial is a perfect square and can be factored as $(\sqrt{a}x + \sqrt{c})^2$ if $b > 0$, or $(\sqrt{a}x - \sqrt{c})^2$ if $b < 0$.

Factoring Perfect Squares

Example

Use the shortcut to verify that $16x^2 - 24x + 9 = (4x - 3)^2$.

$$\begin{aligned} 2\sqrt{16}\sqrt{9} &= 2 \times 4 \times 3 \\ &= 24 \end{aligned}$$

Therefore, since $|b| = 24$ and $b < 0$,

$$\begin{aligned} 16x^2 - 24x + 9 &= (\sqrt{16}x - \sqrt{9})^2 \\ &= (4x - 3)^2 \end{aligned}$$

Factoring Perfect Squares

Example

Factor $81x^2 + 126x + 49$.

It is possible to use decomposition to factor the trinomial, but we would need to find two numbers with a product of 3969 and a sum of 126.

Using the shortcut for perfect squares is a much better solution!

$$\begin{aligned} 2\sqrt{81}\sqrt{49} &= 2 \times 9 \times 7 \\ &= 126 \end{aligned}$$

Therefore, the trinomial is a perfect square and factors as

$$\begin{aligned} 81x^2 + 126x + 49 &= (\sqrt{81}x + \sqrt{49})^2 \\ &= (9x + 7)^2 \end{aligned}$$

Factoring Perfect Squares

Your Turn

Factor $400x^2 - 520x + 169$.

$$\begin{aligned} 2\sqrt{400}\sqrt{169} &= 2 \times 20 \times 13 \\ &= 520 \end{aligned}$$

Therefore, the trinomial is a perfect square and factors as

$$\begin{aligned} 400x^2 - 520x + 169 &= (\sqrt{400}x - \sqrt{169})^2 \\ &= (20x - 13)^2 \end{aligned}$$

Differences of Squares

Sometimes when we multiply two binomials, terms cancel out, producing a binomial instead of a trinomial.

$$\begin{aligned} (3x - 5)(3x + 5) &= 3x \cdot 3x + 3x \cdot 5 - 3x \cdot 5 - 5 \times 5 \\ &= 9x^2 + 15x - 15x - 25 \\ &= 9x^2 - 25 \end{aligned}$$

The resulting binomial is called a *difference of squares*.

Factoring a Difference of Squares

A binomial of the form $ax^2 - c$ can be factored as $(\sqrt{a}x - \sqrt{c})(\sqrt{a}x + \sqrt{c})$.

Factoring Differences of Squares

Example

Factor $25x^2 - 4$.

$\sqrt{25} = 5$ and $\sqrt{4} = 2$, so $25x^2 - 4 = (5x - 2)(5x + 2)$.

Factoring a difference of squares should be quick!

Factoring Differences of Squares

Example

Factor $100x^2 - 289y^2$.

Even with two variables, the binomial is a difference of squares.

$\sqrt{100} = 10$ and $\sqrt{289} = 17$, so $100x^2 - 289y^2 = (10x - 17y)(10x + 17y)$.

Factoring Differences of Squares

Your Turn

Factor $9x^6 - y^4$.

Recall that $x^6 = (x^3)^2$ and $y^4 = (y^2)^2$.

$\sqrt{9} = 3$ and $\sqrt{1} = 1$, so $9x^6 - y^4 = (3x^3 - y^2)(3x^3 + y^2)$.

Factoring Differences of Squares

Your Turn

Factor $x^2 + 36$.

The binomial can not factor as $(x - 6)(x + 6)$, because this would result from $x^2 - 36$.

It can not factor as $(x + 6)(x + 6)$, since this would result from the perfect square $x^2 + 12x + 36$.

It can not factor as $(x - 6)(x - 6)$, since this would result from the perfect square $x^2 - 12x + 36$.

In fact, the binomial (called a *sum of squares*) does not factor at all. If you ever come across a sum of squares, do not even attempt to factor it.

Multi-Stage Factoring

Some polynomials may require more than one level of factoring.

A good strategy is to try to find a *common factor* first, then to analyze the remaining polynomial to see if it it can be factored using another technique.

Multi-Stage Factoring

Example

Factor $6x^3 - 9x^2 - 6x$.

Each term has a common factor of $3x$.

$$6x^3 - 9x^2 - 6x = 3x(2x^2 - 3x - 2)$$

The complex trinomial in the brackets can be factored using decomposition.

$$\begin{aligned} 3x(2x^2 - 3x - 2) &= 3x(2x^2 - 4x + x - 2) \\ &= 3x(2x(x - 2) + 1(x - 2)) \\ &= 3x(x - 2)(2x + 1) \end{aligned}$$

Multi-Stage Factoring

Example

Factor $5x^4 - 80$.

Each term has a common factor of 5.

$$5x^4 - 80 = 5(x^4 - 16)$$

This reveals a difference of squares ...

$$5(x^4 - 16) = 5(x^2 + 4)(x^2 - 4)$$

... and yet *another* difference of squares!

$$5(x^2 - 4)(x^2 + 4) = 5(x^2 + 4)(x - 2)(x + 2)$$

Questions?