MPM2D: Principles of Mathematics

Solving Quadratic Equations

Part 3: The Quadratic Formula

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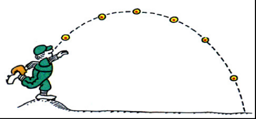


Quadratic Equations

Recap

Solve $2x^2 - 12x + 12 = 0$ by completing the square.

$$\begin{aligned} 2(x^2 - 6x) + 12 &= 0 \\ 2(x^2 - 6x + 9 - 9) + 12 &= 0 \\ 2(x-3)^2 - 6 &= 0 \\ (x-3)^2 &= 3 \\ x - 3 &= \pm\sqrt{3} \\ x &= 3 \pm \sqrt{3} \end{aligned}$$

Quadratic Equations

Consider the general form of a quadratic relation, $y = ax^2 + bx + c$.

We can complete the square to determine values of x for which $ax^2 + bx + c = 0$.

$$\begin{aligned} a\left(x^2 + \tfrac{b}{a}x\right) + c &= 0 \\ a\left(x^2 + \tfrac{b}{a}x + \left(\tfrac{b}{2a}\right)^2 - \left(\tfrac{b}{2a}\right)^2\right) + c &= 0 \\ a\left(\left[x + \tfrac{b}{2a}\right]^2 - \tfrac{b^2}{4a^2}\right) + c &= 0 \\ a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a} + c &= 0 \\ a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a} + \tfrac{4ac}{4a} &= 0 \\ a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2 - 4ac}{4a} &= 0 \end{aligned}$$

Quadratic Equations

Isolate x to find solutions.

$$\begin{aligned} a\left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2 - 4ac}{4a} \\ \left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2 - 4ac}{4a^2} \\ x + \tfrac{b}{2a} &= \pm\sqrt{\tfrac{b^2 - 4ac}{4a^2}} \\ x + \tfrac{b}{2a} &= \pm\tfrac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\ x &= -\tfrac{b}{2a} \pm \tfrac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \tfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Quadratic Formula

The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Quadratic Equations

Example

Verify that the previous equation, $2x^2 - 12x + 12 = 0$, has solutions $x = 3 \pm \sqrt{3}$ using the quadratic formula.

In the equation, $a = 2$, $b = -12$ and $c = 12$.

$$\begin{aligned} x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(12)}}{2(2)} \\ x &= \frac{12 \pm \sqrt{48}}{4} \\ x &= \frac{12 \pm 4\sqrt{3}}{4} \\ x &= \frac{4(3 \pm \sqrt{3})}{4} \\ x &= 3 \pm \sqrt{3} \end{aligned}$$

Quadratic Equations

Example

Solve $x^2 - 6x - 91 = 0$ using the quadratic formula.

In the equation, $a = 1$, $b = -6$ and $c = -91$.

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-91)}}{2(1)} \\ x &= \frac{6 \pm \sqrt{400}}{2} \\ x &= \frac{6 \pm 20}{2} \\ x &= 3 \pm 10 \end{aligned}$$

The two solutions are $x = 3 + 10 = 13$ and $x = 3 - 10 = -7$. The same solutions could have been found by factoring.

Quadratic Equations

Example

Determine the x-intercepts of the parabola defined by $y = -3x^2 - 6x + 15$.

The x-intercepts correspond to the solutions to the quadratic equation $-3x^2 - 6x + 15 = 0$.

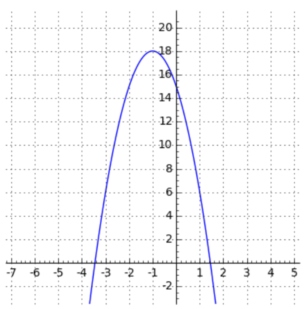
Since moving all three terms to the other side of the equation results in the equation $3x^2 + 6x - 15 = 0$, this equation may also be used. It has the advantage of avoiding a negative denominator.

Quadratic Equations

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(-15)}}{2(3)}$$
$$x = \frac{-6 \pm \sqrt{216}}{6}$$
$$x = \frac{-6 \pm 6\sqrt{6}}{6}$$
$$x = -1 \pm \sqrt{6}$$

The x-intercepts are approximately -3.45 and 1.45.

Quadratic Equations



Quadratic Equations

Example

Solve $2x^2 - x + 5 = 0$.

In the equation, $a = 2$, $b = -1$ and $c = 5$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(5)}}{2(2)}$$
$$x = \frac{1 \pm \sqrt{-39}}{4}$$

At this point there is a problem, since $\sqrt{-39}$ is not a real number.

Therefore, there can be no real solutions to this equation.

The Discriminant

Let $D = b^2 - 4ac$, the expression inside of the radical sign in the quadratic formula.

This expression is known as the *discriminant*, and it can be used to give us information about both the number of real-valued solutions for a quadratic equation.

If $D < 0$, then $\sqrt{D}$ will be undefined for any real numbers.

If $D = 0$, then $\sqrt{D} = 0$, and the quadratic formula will become $x = \frac{-b}{2a}$, for which there is one* real solution.

If $D > 0$, then the quadratic formula is $x = \frac{-b \pm \sqrt{D}}{2a}$, for which there are two distinct real solutions.

The Discriminant

To determine the number of real solutions to a quadratic relation, it is only necessary to calculate the value of the discriminant.

Determining the Number of Real Roots

The number of real solutions to a quadratic equation $ax^2 + bx + c = 0$ can be determined using the discriminant, $D = b^2 - 4ac$.

- If $D < 0$, there are no real solutions.
- If $D = 0$, there is one real solution.
- If $D > 0$, there are two distinct real solutions.

The Discriminant

Example

Determine the number of real solutions to $4x^2 - 20x + 25 = 0$.

Since $D = (-20)^2 - 4(4)(25) = 0$, there is one real solution.

Example

Determine the number of real solutions to $3x^2 + 5x + 4 = 0$.

Since $D = 5^2 - 4(3)(4) = -23$, there are no real solutions.

Questions?