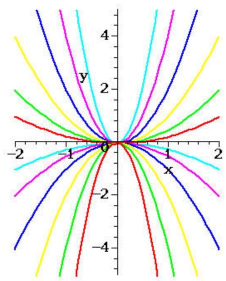
MPM2D: Principles of Mathematics

Completing the Square

Part 1: Simple Trinomials

J. Garvin



Quadratic Relations In Vertex Form

Recap

Sketch a graph of $y = (x+1)^2 - 5$.

The vertex is at $(-1, -5)$.

Since $a = 1$, the step pattern begins $1, 3, 5, \ldots$.

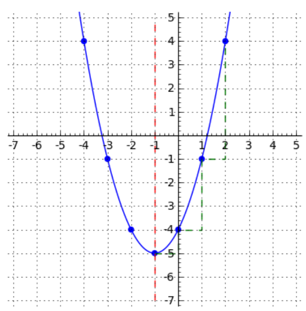
From $(-1, -5)$, move 1 unit right and 1 unit up to $(0, -4)$.

From $(0, -4)$, move 1 unit right and 3 units up to $(1, -1)$.

From $(1, -1)$, move 1 unit right and 5 units up to $(2, 4)$.

Reflect these three points in the axis of symmetry $x = -1$.

Quadratic Relations In Vertex Form



Perfect Square Trinomials

For what value of c will $x^2 + 12x + c$ be a perfect square?

Recall that a perfect square trinomial of the form $x^2 + bx + c$ satisfies the relationship $b = 2\sqrt{c}$. Rearrange and solve for c.

$$\begin{aligned} 12 &= 2\sqrt{c} \\ 6 &= \sqrt{c} \\ c &= 36 \end{aligned}$$

When $c = 36$, the trinomial is $x^2 + 12x + 36$, which factors as $(x+6)^2$.

Perfect Square Trinomials

Generalizing, what is the value of c that will make $x^2 + bx + c$ a perfect square?

$$\begin{aligned} b &= 2\sqrt{c} \\ \frac{b}{2} &= \sqrt{c} \\ c &= \left(\frac{b}{2}\right)^2 \end{aligned}$$

Therefore, if $c = \left(\frac{b}{2}\right)^2$, then $x^2 + bx + c$ is a perfect square.

Completing the Square

One method of converting a quadratic function to vertex form is by a process called *Completing the Square* (CTS).

Note that vertex form, $y = a(x-h)^2 + k$, contains a perfect square, $(x-h)^2$.

To convert from standard form to vertex form, we need to rewrite a trinomial using a perfect square.

To do so with simple trinomials, we can add and subtract a constant term of $\left(\frac{b}{2}\right)^2$. The subtraction ensures that the new relation is equivalent to the original one.

This is the value of c we found earlier.

Completing the Square

Example

Convert $y = x^2 + 6x + 5$ to vertex form.

Since $b = 6$, add and subtract a constant of $\left(\frac{6}{2}\right)^2 = 9$ to the trinomial.

$$y = x^2 + 6x + 9 - 9 + 5$$

The first three terms can now be factored as a perfect square, and the last two terms can be combined.

$$y = (x + 3)^2 - 4$$

Therefore, $y = x^2 + 6x - 5$ can be expressed in vertex form as $y = (x + 3)^2 - 4$.

Completing the Square

Example

What is the minimum value of $y = x^2 + 12x + 16$?

Since $a = 1$, the parabola opens upward, so a minimum value will occur at its vertex.

Since $b = 12$, add and subtract a constant of $\left(\frac{12}{2}\right)^2 = 36$ to the trinomial.

$$y = x^2 + 12x + 36 - 36 + 16$$

The first three terms can now be factored as a perfect square, and the last two terms can be combined.

$$y = (x + 6)^2 - 20$$

Since the vertex is at $(-6, -20)$, the minimum value is -20, occurring when $x = -6$.

Completing the Square

Example

Sketch a graph of $y = x^2 - 8x + 18$

Rather than using a table of values, complete the square to determine the location of the vertex.

Since $b = -8$, add and subtract a constant of $\left(\frac{-8}{2}\right)^2 = 16$ to the trinomial.

$$y = x^2 - 8x + 16 - 16 + 18$$

Factor the first three terms as a perfect square, and combine the last two terms.

$$y = (x - 4)^2 + 2$$

Therefore, $y = x^2 - 8x + 18$ can be expressed in vertex form as $y = (x - 4)^2 + 2$.

Completing the Square

The vertex of the parabola described by $y = (x - 4)^2 + 2$ is at $(4, 2)$.

Since $a = 1$, the step pattern begins $1, 3, 5, \ldots$.

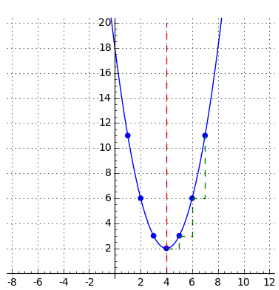
From $(4, 2)$, move 1 unit right and 1 unit up to $(5, 3)$.

From $(5, 3)$, move 1 unit right and 3 units up to $(6, 6)$.

From $(6, 6)$, move 1 unit right and 5 units up to $(7, 11)$.

Reflect these three points in the axis of symmetry $x = 4$.

Completing the Square



Note that the y-intercept is at $(0, 18)$, as given by the original equation $y = x^2 - 8x + 18$.

Questions?