Transformations

In most cases, the graph of a function is similar to a simpler version, but may appear stretched, shifted or reflected to some extent.

The simplest version of a function that possesses all of the same characteristics of the derived function is called a parent function or a base function.

If we know information about a particular base function, it may be possible to sketch a graph of the derived function by analyzing the transformations that have been applied to the base function.

Transformations of Polynomial Functions

A polynomial of the form \( f(x) = a(b(x - c))^n + d \), where \( a, b, c \) and \( d \) are real constants, and \( n \) is a natural number, is a transformation of some power function \( g(x) = x^n \).

In the form above:
- \( a \) is a vertical stretch/compression, and possibly a reflection.
- \( b \) is a horizontal stretch/compression, and possibly a reflection.
- \( c \) is a horizontal translation.
- \( d \) is a vertical translation.

Vertical Stretches/Compressions

Example

Sketch graphs of \( f(x) = 2x^3 \) and \( g(x) = \frac{1}{3}x^3 \).

For \( f(x) \), \( |a| > 1 \), so it has a vertical stretch by a factor of 2. All points are twice as far from the x-axis as they are on the graph of \( y = x^3 \).

For \( g(x) \), \( 0 < |a| < 1 \), so it has a vertical compression by a factor of \( \frac{1}{3} \). All points are one-third as far from the x-axis as they are on the graph of \( y = x^3 \).
Vertical Reflections
If \( a < 0 \), then a transformed power function has undergone a vertical reflection (reflection in the \( x \)-axis).

Horizontal Stretches/Compressions
Example
Sketch graphs of \( f(x) = (3x)^3 \) and \( g(x) = (\frac{1}{2}x)^3 \).

For \( f(x) \), \(|b| > 1\), so it has a horizontal compression by a factor of \( \frac{1}{3} \). All points are three times as far from the \( f(x) \)-axis as they are on the graph of \( y = x^3 \).

For \( g(x) \), \( 0 < |b| < 1 \), so it has a horizontal stretch by a factor of 2. All points are twice as far from the \( f(x) \)-axis as they are on the graph of \( y = x^3 \).

Horizontal Reflections
If \( b < 0 \), then a transformed power function has undergone a horizontal reflection (reflection in the \( f(x) \)-axis).

Vertical and Horizontal Translations
Example
Sketch a graph of \( f(x) = (x - 2)^3 + 3 \).

The graph of \( f(x) \) has two transformations: a horizontal translation 2 units to the right, and a vertical translation 3 units up.

Neither transformation affects the shape of the graph, only its position.
**Identifying Transformations From an Equation**

**Example**
Identify the base function, and the transformations applied to it, to create the function $f(x) = 2(3x - 1)^3 - 5$.

The base function is $y = x^3$.
The 2 indicates a vertical stretch by a factor of 2.
The 3 indicates a horizontal compression by a factor of $\frac{1}{3}$.
There is a horizontal translation $\frac{1}{3}$ of a unit to the right, since the equation can be written $f(x) = 2(3(x - \frac{1}{3}))^3 - 5$.
Finally, there is a vertical translation down 5 units.

**Graphing Transformed Functions**

**Example**
Sketch a graph of $f(x) = -2(x - 1)^4 + 3$.

The base power function, $y = x^4$, has Q2-Q1 end behaviour and its “vertex” at the origin.
$f(x)$ has a vertical reflection, so its end behaviour is Q3-Q4.
There is a vertical stretch by a factor of 2, a horizontal translation 1 unit to the right, and a vertical translation 3 units up.

**Determining Equations From Graphs**

**Example**
Determine an equation for the function shown below.

The function has Q2-Q4 end behaviour, so it has an odd degree (likely cubic) and negative leading coefficient.
The “pivot point” of the function is at (2,4), indicating a vertical translation up 4 units and a horizontal translation right 2 units.
To determine if a vertical stretch has occurred, note that the function has an $f(x)$-intercept at 6.
To go from (2, 4) to (0, 6), there is a vertical change of 2 for a horizontal change of 2.
For the parent function $y = x^3$, there is a horizontal change of 2 from (0, 0) to (2, 8), resulting in a vertical change of 8.
Thus, there is a vertical compression by a factor of $\frac{1}{4}$.
A possible equation, then, is $f(x) = -\frac{1}{4}(x - 2)^3 + 4$.

**Questions?**