MHF4U: Advanced Functions

Reciprocals of Quadratic Functions

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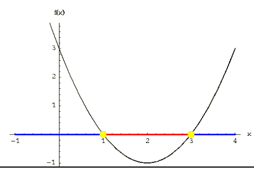


Reciprocals of Quadratic Functions

A quadratic function has the form $f(x) = ax^2 + bx + c$ in standard form, where a, b and c are real coefficients.

What does the graph of the *reciprocal* of a quadratic look like?

There are three cases to consider, depending on the factorability of the quadratic.

Asymptotes

Vertical asymptotes occur when the denominator of a rational expression is zero.

Thus, the roots of a quadratic expression in the denominator correspond to any vertical asymptotes.

Since a quadratic may have zero, one or two real roots, the reciprocal of a quadratic may have zero, one or two vertical asymptotes.

Like reciprocals of linear functions, horizontal asymptotes can be determined by dividing each term by the highest power, then evaluating as $x \to \infty$.

Asymptotes

Example

Determine the equations of any asymptotes for $f(x) = \dfrac{1}{x^2 - 4}$.

After factoring, $f(x) = \dfrac{1}{(x-2)(x+2)}$.

There are two vertical asymptotes: one with equation $x = -2$, and the other $x = 2$.

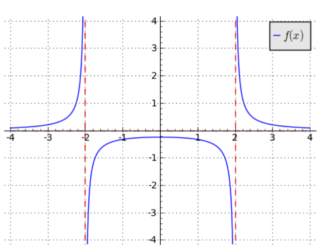
Divide the expression by x^2 and let $x \to \infty$.

$$\frac{\frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}} = \frac{0}{1-0} = 0$$

The equation of the horizontal asymptote is $f(x) = 0$.

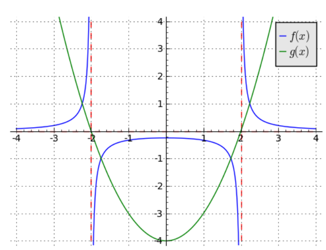
Asymptotes

A graph of $f(x) = \dfrac{1}{x^2 - 4}$ is below.



How does the graph of $f(x)$ compare to that of $g(x) = x^2 - 4$?

Intercepts



$f(x)$ is symmetric about the same axis as $g(x)$.

A local maximum occurs on $f(x)$ where there is a local minimum on $g(x)$.

Intercepts

As with any function, the $f(x)$-intercept can be found by substituting $x = 0$ into its equation.

x-intercepts will occur when the numerator evaluates to zero.

If the reciprocal of a quadratic has the form $f(x) = \frac{1}{ax^2 + bx + c}$, then there will always be a horizontal asymptote at $f(x) = 0$.

Verifying the last example, the $f(x)$-intercept is at $\frac{1}{0^2-4} = -\frac{1}{4}$ and there are no x-intercepts.

Minima/Maxima

Since functions of the form $f(x) = \frac{1}{ax^2 + bx + c}$ have line symmetry, any minimum or maximum point will occur halfway between the two vertical asymptotes.

Substituting in this middle value allows us to determine the coordinate where there is a local min/max.

In the previous example, the vertical asymptotes were at $x = -2$ and $x = 2$.

Therefore, a local minimum or maximum will occur when $x = \frac{-2+2}{2} = 0$, or at $\left(0, -\frac{1}{4}\right)$.

How do we determine if the point is a local minimum or maximum?

Minima/Maxima

Example

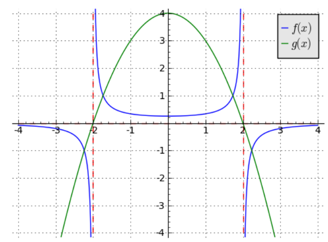
Determine any local minima/maxima for $f(x) = -\frac{1}{x^2 - 4}$.

This example is the same as the previous one, except that there has been a vertical reflection.

This will have the effect of changing the local maximum to a local minimum.

When there are two vertical asymptotes, a function of the form $f(x) = \frac{k}{ax^2 + bx + c}$ will have a local minimum when $k < 0$ and a local maximum when $k > 0$.

Minima/Maxima



Sketching Graphs

Example

Sketch a graph of $f(x) = \frac{1}{x^2 - 4x + 3}$, and state its domain and range.

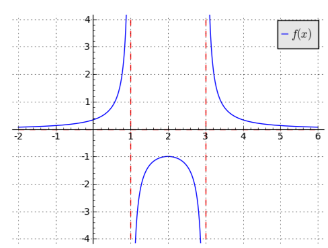
$f(x) = \frac{1}{(x-1)(x-3)}$, so there are vertical asymptotes at $x = 1$ and $x = 3$.

There is a horizontal asymptote at $f(x) = 0$, and there are no x-intercepts.

The $f(x)$-intercept occurs at $\frac{1}{0^2 - 4(0) + 3} = \frac{1}{3}$.

Since $k > 0$, a local maximum will occur when $x = 2$, or at $(2, -1)$.

Sketching Graphs



The domain is $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$ and the range is $(-\infty, -1] \cup (0, \infty)$.

Sketching Graphs

Example

Sketch a graph of $f(x) = -\dfrac{1}{x^2 - 6x + 9}$.

$f(x) = -\dfrac{1}{(x-3)^2}$, a perfect square, so there is a single vertical asymptote at $x = 3$.

There is a horizontal asymptote at $f(x) = 0$, and there are no x-intercepts.

The $f(x)$-intercept occurs at $-\frac{1}{0^2 - 6(0) + 9} = -\frac{1}{9}$.

How about the local minimum/maximum?

Sketching Graphs

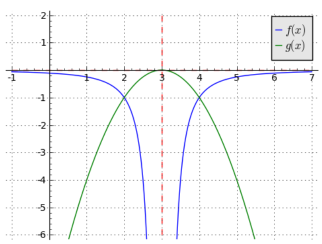
Test values on either side of the asymptote to determine whether the function is positive or negative.

$f(2) = -\frac{1}{2^2 - 6(2) + 9} = -1$, and $f(4) = -\frac{1}{4^2 - 6(4) + 9} = -1$.

Since the function is negative on either side of the asymptote, then as $x \to 3$ from the left, $f(x) \to -\infty$, and as $x \to 3$ from the right, $f(x) \to -\infty$.

Therefore, there is no local minimum or maximum, as $f(x)$ decreases without limit.

Sketching Graphs



A reciprocal of a quadratic with one vertical asymptote will always have this shape, possibly reflected vertically.

Sketching Graphs

Example

Sketch a graph of $f(x) = \dfrac{1}{x^2 + 1}$.

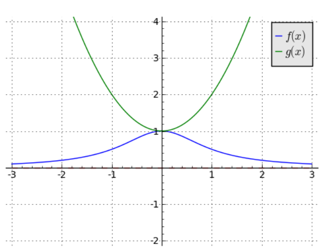
$f(x)$ does not factor, so there are no vertical asymptotes.

There is a horizontal asymptote at $f(x) = 0$, and there are no x-intercepts.

The $f(x)$-intercept occurs at $\frac{1}{0^2 + 1} = 1$.

Since the $f(x)$-intercept is positive, the function lies completely above the horizontal asymptote.

Sketching Graphs



A reciprocal of a quadratic with no vertical asymptote will always have this shape, possibly reflected vertically.

Questions?