MHF4U: Advanced Functions

Rational Functions of the Form $\frac{ax+b}{cx+d}$

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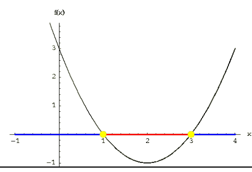


Rational Functions

Recall that a rational function is a ratio of two polynomial functions, $p(x)$ and $q(x)$, such that $f(x) = \frac{p(x)}{q(x)}$.

Since $q(x) \neq 0$, there will often be some form of discontinuity, such as an asymptote of a hole.

In this section, we will investigate rational functions that have the form $f(x) = \frac{ax+b}{cx+d}$.

Such functions have properties that are predictable, lending them to easy-to-draw graphs.

Rational Functions

Example

Graph the function $f(x) = \frac{x+4}{x-2}$ and describe its properties.

There is a vertical asymptote at $x = 2$.

Divide each term by x to find the equation of the horizontal asymptote.

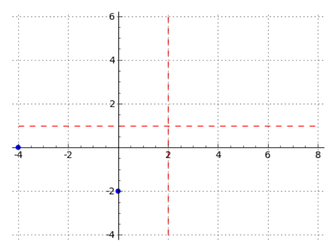
$$\frac{\frac{x}{x}+\frac{4}{x}}{\frac{x}{x}-\frac{2}{x}} = \frac{1+0}{1-0}$$
$$= 1$$

A horizontal asymptote occurs at $f(x) = 1$.

The x-intercept is at $x = -4$ and the $f(x)$-intercept is at -2.

Rational Functions

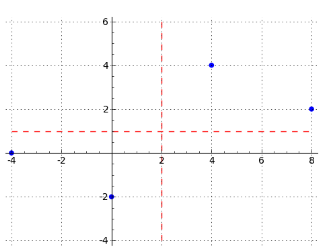
Putting things together, we obtain the following graph.



To determine the function's behaviour to the right of the vertical asymptote, test values of x greater than 2.

Rational Functions

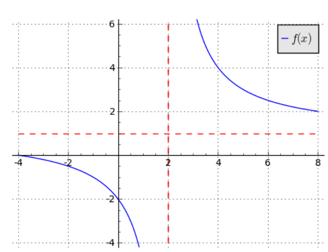
$f(4) = 4$ and $f(8) = 2$, resulting in the following graph.



From the graph, it appears that the function is symmetric about the asymptotes.

Rational Functions

A complete graph of $f(x)$ is shown below, confirming the symmetry.



Rational Functions

Example

Graph the function $f(x) = \frac{2x-1}{x+1}$.

There is a vertical asymptote at $x = -1$.

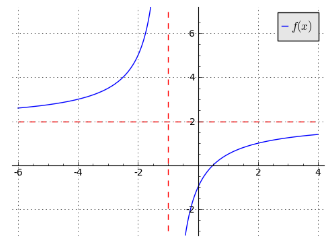
Divide each term by x to find the equation of the horizontal asymptote.

$$\frac{\frac{2x}{x} - \frac{1}{x}}{\frac{x}{x} + \frac{1}{x}} = \frac{2-0}{1+0}$$
$$= 2$$

A horizontal asymptote occurs at $f(x) = 2$.

The x-intercept is at $x = \frac{1}{2}$ and the $f(x)$-intercept is at -1.

Rational Functions

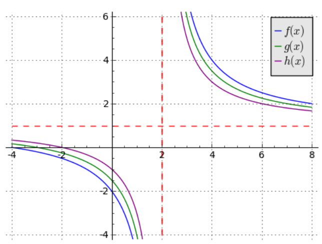


Since the function is symmetric about the asymptotes, the intercepts have image points at $(-2, 5)$ and $\left(-\frac{5}{2}, 4\right)$.

Rational Functions

Example

Compare the graphs of $f(x) = \frac{x+4}{x-2}$, $g(x) = \frac{x+3}{x-2}$ and $h(x) = \frac{x+2}{x-2}$ below.



Rational Functions

All three functions have the form $\frac{ax+b}{cx+d}$.

As the value of b increases, the function is stretched further from the asymptotes.

The value of b has no effect on the vertical and horizontal asymptotes.

Rational Functions

Example

Determine the equation of a rational function with the following features:

- a vertical asymptote at $x = 3$
- a horizontal asymptote at $f(x) = 2$
- an x-intercept at $x = 1$

Rational Functions

There are many functions that can satisfy these conditions.

Since a vertical asymptote occurs at $x = 3$, let the denominator be $x - 3$.

In order for a horizontal asymptote to occur at $f(x) = 2$, and since $c = 1$, the value of a must be 2, since $\frac{a}{c} = \frac{2}{1} = 2$.

The x-intercept occurs when the numerator is zero, or $2x + b = 0$. Isolating x, this becomes $x = -\frac{b}{2}$.

Since the x-intercept is 1, $-\frac{b}{2} = 1$, or $b = -2$.

Thus, a possible equation is $f(x) = \frac{2x-2}{x-3}$.

Questions?