MHF4U: Advanced Functions

Properties of Logarithms

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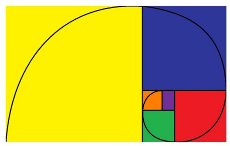
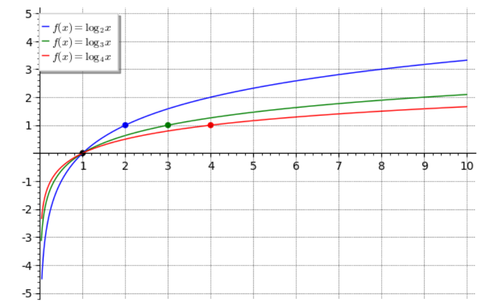


Basic Properties of Logarithms

Consider the graph of the logarithmic functions below.



Basic Properties of Logarithms

From the graphs, we can identify some basic properties of logarithms.

Basic Properties of Logarithms

- $\log_b b = 1$
- $\log_b 1 = 0$
- $\log_b 0$ is undefined

These are useful when simplifying logarithmic expressions.

Basic Properties of Logarithms

Example

Evaluate $\log_7 1$.

Since $\log_b 1 = 0$, $\log_7 1 = 0$.

Example

Evaluate $3 \log_9 9$.

Since $\log_b b = 1$, $3 \log_9 9 = 3(1) = 3$.

Power Law of Logarithms

Evaluate each pair of expressions, and make a prediction.

$3 \log 100$ and $\log 100^3$ $\qquad$ $2 \log_3 9$ and $\log_3 9^2$

$$\begin{aligned} 3 \log 100 &= 3(2) \\ &= 6 \end{aligned} \qquad \begin{aligned} \log 100^3 &= \log 1\,000\,000 \\ &= 6 \end{aligned}$$

$$\begin{aligned} 2 \log_3 9 &= 2(2) \\ &= 4 \end{aligned} \qquad \begin{aligned} \log_3 9^2 &= \log_3 81 \\ &= 4 \end{aligned}$$

Pairs have the same value, suggesting $k \log_b x = \log_b x^k$.

Power Law of Logarithms

We can show that this is true for all logarithmic expressions. Let $\log_b x$ be some value v.

$$\begin{aligned} v &= \log_b x \\ b^v &= x \\ (b^v)^n &= x^n \\ b^{nv} &= x^n \\ \log_b x^n &= nv \\ \log_b x^n &= n \log_b x \end{aligned}$$

Power Law of Logarithms

For all $b > 0$, $b \neq 1$, and $x > 0$, $\log_b x^n = n \log_b x$.

Power Law of Logarithms

Example

Evaluate $\log_2 8^7$.

Since $8^7 = 2\,097\,152$, evaluating will be difficult using a trial-and-error approach. Using the power law of logarithms should make things easier.

$$\begin{aligned}\log_2 8^7 &= 7\log_2 8\\ &= 7(3)\\ &= 21\end{aligned}$$

Power Law of Logarithms

Example

Evaluate $\log_5 25^{-3}$.

Since $25^{-3} = 0.000064$ is an awkward decimal value, and $25^{-3} = \frac{1}{15\,625}$ is not much better as a fraction, using the power law is useful in this case as well.

$$\begin{aligned}\log_5 25^{-3} &= -3\log_5 25\\ &= -3(2)\\ &= -6\end{aligned}$$

Power Law of Logarithms

Example

Simplify the expression $\log_b b^n$.

We can use the power law of logarithms, as well as some basic properties, to simplify this expression.

$$\begin{aligned}\log_b b^n &= n\log_b b\\ &= n(1)\\ &= n\end{aligned}$$

Change of Base Formula

Most modern calculators will allow you to specify the base of a logarithmic function, but some only have buttons for logarithms with base 10 (log) and base e (ln). Some programming-related calculators may have a button for base 2, but these are rare.

In order to perform calculations with a base other than 10 or e, we must find a way to express a given logarithmic function using either of these two bases.

Fortunately, there is a formula for this.

Change of Base Formula

$$\begin{aligned}\log_b x &= v\\ b^v &= x\\ \log_k b^v &= log_k x\\ v\log_k b &= log_k x\\ v &= \frac{\log_k x}{\log kb}\\ \log_b x &= \frac{\log_k x}{\log_k b}\end{aligned}$$

Change of Base Formula

By choosing a convenient base k, like $k = 10$, we obtain a method for calculating $\log_b x$ when it is difficult to work in terms of base b.

Change of Base Formula For Logarithms

$\log_b x = \frac{\log x}{\log b}$, where $b > 0$, $b \neq 1$, and $x > 0$.

Sometimes this formula is expressed using the *natural logarithm*, $\log_e x = \ln x$, as $\log_b x = \frac{\ln x}{\ln b}$. We may examine the number e, and some of its properties, in a later lesson.

Change of Base Formula

Example

Evaluate $\log_5 100$.

Since $5^2 = 25$ and $5^3 = 125$, the value of $\log_5 100$ must be somewhere between 2 and 3. Use the change of base formula to express it in terms of base 10 instead.

$$\begin{aligned}\log_5 100 &= \frac{\log 100}{\log 5} \\ &\approx 2.86\end{aligned}$$

Change of Base Formula

Example

Evaluate $\log_{100} 1\,000$.

Since $100^1 = 100$ and $100^2 = 10\,000$, the value of $\log_{100} 1\,000$ must be somewhere between 1 and 2. This example will not require the use of a calculator, since we can rewrite both 100 and 1 000 as powers of 10.

$$\begin{aligned}\log_{100} 1\,000 &= \frac{\log 1\,000}{\log 100} \\ &= \frac{\log 10^3}{\log 10^2} \\ &= \frac{3 \log 10}{2 \log 10} \\ &= \frac{3}{2}\end{aligned}$$

Change of Base Formula

In the last example, our answer of $\frac{3}{2}$ was exactly halfway between 1 and 2, as we predicted; however, 1 000 is *not* halfway between 100 and 10 000.

This is because logarithmic functions do not increase linearly (much in the same way that sinusoidal functions do not increase linearly either).

This is important to remember. For example, $2^4 = 16$ and $2^5 = 32$, but $\log_2 24 \neq 4.5$ (or $2^{4.5} \neq 24$). Rather, $\log_2 24 \approx 4.585$.

Product and Quotient Laws of Logarithms

Evaluate each pair of expressions, and make a prediction.

$\log(3 \times 5)$ and $\log 3 + \log 5$ $\quad$ $\log\left(\frac{5}{3}\right)$ and $\log 5 - \log 3$

$$\begin{aligned}\log(3 \times 5) &= \log(15) \\ &\approx 1.176\end{aligned} \qquad \begin{aligned}\log 3 + \log 5 &\approx 0.477 + 0.699 \\ &\approx 1.176\end{aligned}$$

$$\begin{aligned}\log\left(\tfrac{5}{3}\right)& \\ &\approx 0.222\end{aligned} \qquad \begin{aligned}\log 5 - \log 3 &\approx 0.699 - 0.477 \\ &\approx 0.222\end{aligned}$$

Like before, pairs have the same value.

Product and Quotient Laws of Logarithms

The previous examples illustrate the product and quotient laws for logarithms.

Product and Quotient Laws of Logarithms

For all $b > 0$, $b \neq 1$, $m > 0$ and $n > 0$:

- $\log_b(m \times n) = \log_b m + \log_b n$
- $\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$

Product and Quotient Laws of Logarithms

Example

Express $\log_5 7 + \log_5 9$ as a single logarithm.

Since the two logarithms have a common base of 5, we can use the product rule.

$$\begin{aligned}\log_5 7 + \log_5 9 &= \log_5(7 \times 9) \\ &= \log_5 63\end{aligned}$$

Product and Quotient Laws of Logarithms

Example

Evaluate $\log_3 72 - \log_3 8$.

Since the two logarithms have a common base of 3, we can use the quotient rule.

$$\begin{aligned}\log_3 72 - \log_3 8 &= \log_5\left(\tfrac{72}{8}\right)\\ &= \log_3 9\\ &= 2\end{aligned}$$

Product and Quotient Laws of Logarithms

Example

If $u = \log 12$ and $v = \log 2$, express $\log 24$ and $\log 3$ in terms of u and v.

Since $12 \times 2 = 24$,
$\log 24 = \log(12 \times 2) = \log 12 + \log 2 = u + v$.

Expressing $\log 3$ in terms of u and v is a bit more involved.

$$\begin{aligned}\log 3 &= \log\left(\tfrac{1}{2} \times \tfrac{12}{2}\right)\\ &= \log \tfrac{1}{2} + \log \tfrac{12}{2}\\ &= \log 1 - \log 2 + \log 12 - \log 2\\ &= 0 - v + u - v\\ &= u - 2v\end{aligned}$$

Questions?