Average Rate of Change

Recap
A particle moves in a straight line, according to the equation \( d(t) = -2t^3 + 5t - 1 \), where \( d \) is the distance, in metres, after \( t \) seconds. Determine the average rate of change between the third and seventh seconds.

Calculate \( d(3) \) and \( d(7) \).

\[
d(3) = -2(3)^3 + 5(3) - 1 = -40
\]

\[
d(7) = -2(7)^3 + 5(7) - 1 = -652
\]

Thus, the slope is \( \frac{-652 - (-40)}{7 - 3} = -153 \text{ m/s} \).

Instantaneous Rate of Change

Recall that as the width of the interval decreases, the slope of a secant approaches that of a tangent at a given point.

If we wish to estimate the instantaneous rate of change from a graph, we can approximate the slope of the tangent at a specific point by using one of two methods:

- by using the specific point and another nearby point on the graph, we can create a small interval,
- drawing a tangent to the graph as best as possible and using a second point on the tangent.

These two methods may produce different results, depending on the values used.

Example
Estimate the instantaneous rate of change for the function below when \( x = 1 \), using the nearby point \((2, 5)\).

Use the points \((1, 0)\) and \((2, 5)\), both on the graph, to find the slope of the secant.

\[
\text{slope} = \frac{5 - 0}{2 - 1} = 5
\]

Based on the selected interval, the instantaneous rate of change is estimated to be 5.

Example
Estimate the instantaneous rate of change for the same function when \( x = 1 \), using the point \((3, 8)\) on the tangent.
Instantaneous Rate of Change

Use the point \((1, 0)\) on the graph, and \((3, 8)\) on the tangent.

\[
slope = \frac{8 - 0}{3 - 1} = 4
\]

Using these two points, the instantaneous rate of change is estimated to be 4.

**Example**

Estimate the instantaneous rate of change at \(x = 2\) for the function described by the table of values below.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>-11</td>
<td>-3</td>
<td>3</td>
<td>7</td>
<td>17</td>
<td>39</td>
</tr>
</tbody>
</table>

**Difference Quotient**

The slope of a secant for a given function, \(f(x)\), on the interval \([a, a+h]\), for some real values \(a\) and \(h\), is given by

\[
slope = \frac{f(a+h) - f(a)}{h}
\]

The difference quotient, when used with the idea of limits, forms the basis of many fundamental rules of differential calculus.

**Instantaneous Rate of Change**

Sometimes, data is provided in a table of values, rather than using a graph.

The same technique can be used as before – find a small interval, and calculate the slope of the secant.

**Example**

Estimate the instantaneous rate of change at \(x = 2\) for the function described by the table of values below.

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Using the points \((2, 3)\) and \((3, 7)\) from the table, we can estimate the instantaneous rate of change.

\[
slope = \frac{7 - 3}{3 - 2} = 4
\]

Note that we could have also selected the points \((1, -3)\) and \((2, 3)\) instead.

\[
slope = \frac{3 - (-3)}{2 - 1} = 6
\]

Recall that the slope of a secant to a given function is given by "rise over run".

When \(x = a\), for some real value \(a\), then the value of the function is \(f(a)\).

If the secant spans some interval with width \(h\), then the value of the function at \(x = a + h\) is \(f(a + h)\).

Thus, using "rise over run",

\[
slope = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}
\]

This is an important formula, and is often referred to as the difference quotient.
Instantaneous Rate of Change

Example

Estimate the instantaneous rate of change for the function
\[ f(x) = 3x^2 - 4x + 1 \] when \( x = 1 \).

Using a very small interval, say \([1, 1.0001]\), should give a
good approximation of the instantaneous rate of change
when \( x = 1 \). In this case, \( a = 1 \) and \( h = 0.0001 \).

\[
\begin{align*}
 f(1) &= 3(1)^2 - 4(1) + 1 \\
 &= 0 \\
 f(1.0001) &= 3(1.0001)^2 - 4(1.0001) + 1 \\
 &= 0.00020003
\end{align*}
\]

This value is very close to 2, which is the instantaneous rate
of change when \( x = 1 \).

You will learn various techniques for determining
instantaneous rates of change without using the difference
quotient in any calculus class.

Use the difference quotient to estimate the rate of change.
\[
\frac{0.00020003 - 0}{0.0001} \approx 2.0003
\]

A graph of the tangent to \( f(x) = 3x^2 - 4x + 1 \) at \( x = 1 \) is
below.

Questions?