MHF4U: Advanced Functions

Exact Values of Trigonometric Ratios

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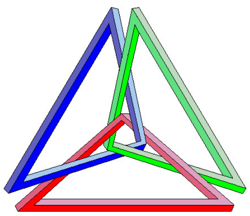


Exact Values of Trigonometric Ratios

Recall the three primary trigonometric ratios, and the three secondary trigonometric ratios for a right triangle:

$\sin A = \dfrac{\text{opp}}{\text{hyp}}$, $\cos A = \dfrac{\text{adj}}{\text{hyp}}$, $\tan A = \dfrac{\text{opp}}{\text{adj}}$,

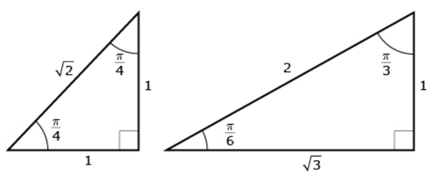
$\csc A = \dfrac{\text{hyp}}{\text{opp}}$, $\sec A = \dfrac{\text{hyp}}{\text{adj}}$, $\cot A = \dfrac{\text{adj}}{\text{opp}}$.

In the past, we have seen exact values for special angles expressed in degrees ($30°, 45°$ and $60°$).

We can also express these angles using radians $\left(\frac{\pi}{6}, \frac{\pi}{4} \text{ and } \frac{\pi}{3}\right)$, producing the same exact values.

Exact Value of a Trigonometric Ratio

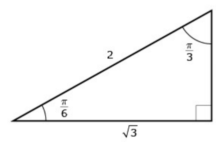
One way to derive the exact values for $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ is to use the following right triangles.



Exact Value of a Trigonometric Ratio

Example

Determine the exact value of $\sin \frac{\pi}{3}$.



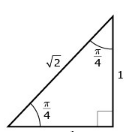
Using the 1-2-$\sqrt{3}$ triangle, the opposite side to $\frac{\pi}{3}$ has a length of $\sqrt{3}$, while the hypotenuse has a length of 2.

Therefore, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Exact Value of a Trigonometric Ratio

Example

Determine the exact value of $\sec \frac{\pi}{4}$.

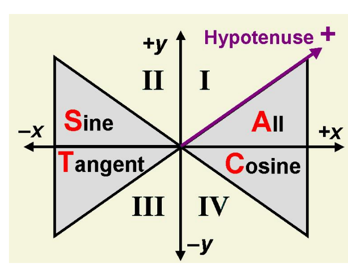


Using the 1-1-$\sqrt{2}$ triangle, the hypotenuse has a length of $\sqrt{2}$ and the adjacent side to $\frac{\pi}{4}$ has a length of 1.

Therefore, $\sec \frac{\pi}{4} = \sqrt{2}$.

Trigonometric Ratios In Quadrants 2, 3 and 4

If the terminal arm of an angle falls in quadrants 2, 3 or 4, we use a *reference angle* of $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ in conjunction with the CAST rule to determine the value and the sign.

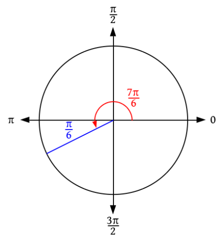


Trigonometric Ratios In Quadrants 2, 3 and 4

Example

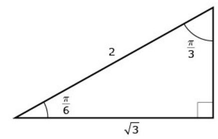
Determine the exact value of $\cos \frac{7\pi}{6}$.

An angle of $\frac{7\pi}{6}$ lies in quadrant 3, with reference angle $\frac{\pi}{6}$.



Trigonometric Ratios In Quadrants 2, 3 and 4

Since cosine is negative in quadrant 3, the exact value will be negative.



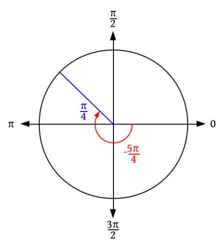
Therefore, $\cos \frac{7\pi}{6} = -\frac{\text{adj}}{\text{hyp}} = -\frac{\sqrt{3}}{2}$.

Trigonometric Ratios In Quadrants 2, 3 and 4

Example

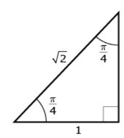
Determine the exact value of $\cot\left(-\frac{5\pi}{4}\right)$.

An angle of $-\frac{5\pi}{4}$ lies in quadrant 2, with reference angle $\frac{\pi}{4}$.



Trigonometric Ratios In Quadrants 2, 3 and 4

Since tangent (and cotangent) is negative in quadrant 2, the exact value will be negative.



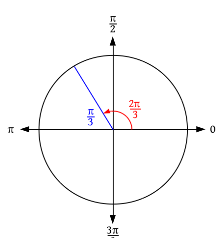
Therefore, $\cot\left(-\frac{5\pi}{4}\right) = -\frac{\text{adj}}{\text{opp}} = -1$.

Trigonometric Ratios In Quadrants 2, 3 and 4

Example

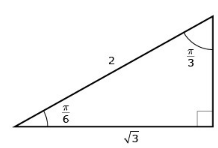
Determine the exact value of $\csc^2 \frac{2\pi}{3}$.

An angle of $\frac{2\pi}{3}$ lies in quadrant 2, with reference angle $\frac{\pi}{3}$.



Trigonometric Ratios In Quadrants 2, 3 and 4

Since sine (and cosecant) is positive in quadrant 2, the exact value will be positive.



Therefore, $\csc \frac{2\pi}{3} = \frac{\text{hyp}}{\text{opp}} = \frac{2}{\sqrt{3}}$.

Thus, $\csc^2 \frac{2\pi}{3} = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$.

Working with Trigonometric Ratios

Example

Determine the exact value of $\cos\frac{\pi}{4}\cdot\sin\frac{5\pi}{4}+\cos\frac{5\pi}{3}\cdot\sin\frac{2\pi}{3}$.

$$\begin{aligned}\cos\frac{\pi}{4}\cdot\sin\frac{5\pi}{4}+\cos\frac{5\pi}{3}\cdot\sin\frac{2\pi}{3} &= \frac{1}{\sqrt{2}}\cdot\frac{-1}{\sqrt{2}}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\\ &= -\frac{1}{2}+\frac{\sqrt{3}}{4}\\ &= \frac{\sqrt{3}-2}{4}\end{aligned}$$

Working with Trigonometric Ratios

Example

Determine the exact value of $\csc\frac{\pi}{6}\left(\sec\frac{7\pi}{4}+\cot\frac{2\pi}{3}\right)$.

$$\begin{aligned}\csc\frac{\pi}{6}\left(\sec\frac{7\pi}{4}+\cot\frac{2\pi}{3}\right) &= 2\left(\sqrt{2}-\frac{1}{\sqrt{3}}\right)\\ &= 2\left(\frac{\sqrt{6}-1}{\sqrt{3}}\right)\\ &= \frac{2\sqrt{6}-2}{\sqrt{3}}\\ &= \frac{2\sqrt{18}-2\sqrt{3}}{3}\\ &= \frac{6\sqrt{2}-2\sqrt{3}}{3}\end{aligned}$$

Working with Trigonometric Ratios

Example

A 5 m ladder leans against a wall at an angle of $\frac{\pi}{4}$ to the ground. If the ladder is raised so that it makes an angle of $\frac{\pi}{3}$ with the ground, what is the change in height?

At an angle of $\frac{\pi}{4}$, the height, h, of the ladder is given by $\sin\frac{\pi}{4}=\frac{h}{5}$.

Therefore, $h=5\sin\frac{\pi}{4}=\frac{5\sqrt{2}}{2}$.

Similarly, when the ladder makes an angle of $\frac{\pi}{3}$, the height is given by $\sin\frac{\pi}{3}=\frac{h}{5}$, so $h=5\sin\frac{\pi}{3}=\frac{5\sqrt{3}}{2}$.

The change in height, then, is $\frac{5\sqrt{3}}{2}-\frac{5\sqrt{2}}{2}$, or $\frac{5(\sqrt{3}-\sqrt{2})}{2}$ m.

Questions?