MHF4U: Advanced Functions

Descartes Rule of Signs

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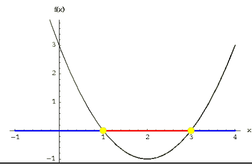


Solving Polynomial Equations

Recap

Determine the values of any real solutions to the equation $x^3 - 9x^2 + 28x - 30 = 0$.

By the IZT, potential factors are ±1, ±2, ±3, ±5, ±6, ±10, ±15 and ±30.

Since $3^3 - 9(3)^2 + 28(3) - 30 = 0$, $x - 3$ is a factor.

	1	−9	28	−30
3		3	−18	30
	1	−6	10	0

The quotient, $x^2 - 6x + 10$, does not factor, so there is one real solution, $x = 3$.

Descartes' Rule of Signs

Using the IZT or RZT, we can identify possible solutions to a given polynomial equation using a smaller sample space than by purely guessing.

If there are many possibilities (the last example had 16), it may be useful to predict just *how many* real solutions there might be.

A technique that does exactly this is called *Descartes' Rule of Signs*. It allows us to obtain an upper bound on the number of possible real solutions (either positive or negative).

Descartes' Rule of Signs

Consider the equation $x^2 - 5x + 6 = 0$.

This factors as $(x - 2)(x - 3) = 0$ and has two positive solutions, $x = 2$ and $x = 3$, both of which are factors of the constant value 6.

Note that in the equation, the signs of the terms alternate between positive and negative.

In this case, there are two sign changes (positive to negative, then negative to positive).

Descartes' Rule of Signs

Now consider the equation $2x^2 + 7x + 3 = 0$.

This factors as $(2x + 1)(x + 3) = 0$ and has two negative solutions, $x = -\frac{1}{2}$ and $x = -3$, both of which are ratios of the factors of the constant value 3 and leading coefficient 2.

In this equation, there are no sign changes.

However, if we were to substitute all values of x with its negative counterpart, $-x$, we would obtain the following.

$$2(-x)^2 + 7(-x) + 3 = 0$$
$$2x^2 - 7x + 3 = 0$$

Note that with this substitution, there are two sign changes.

Descartes' Rule of Signs

Finally, consider the equation $x^3 - 7x^2 + 12x - 10 = 0$.

This factors as $(x - 5)(x^2 - 2x + 2)$, where the quadratic is non-factorable. There is one positive real solution, $x = 5$.

There are three sign changes in the original equation, and none in the equation where x is replaced by $-x$.

$$-x^3 - 7x^2 - 12x - 10 = 0$$

In this case, the relationship between the signs and the solutions is not so clear.

Descartes' Rule of Signs

Descartes' Rule of Signs is a technique in which each sign change from one term to the next indicates a possible solution to a polynomial equation.

A potential positive real solution occurs when there is a sign change between the terms of the original polynomial expression, while a potential negative real solution occurs when there is a sign change in the expression where x is replaced by $-x$.

Complex (non-real) solutions will always occur in pairs, so if a polynomial equation has 5 possible solutions, then either 4 of them, 2 of them, or 0 of them may be complex.

Thus, when counting the possible number of real solutions, it is necessary to count backward by twos.

Descartes' Rule of Signs

Example

Determine the maximum possible number of real solutions to the equation $x^3 + 2x^2 - 5x + 10 = 0$.

Examining the signs of the terms, there are two sign changes: positive to negative between the second and third terms, and negative to positive between the third and fourth terms.

Therefore, there are either 2 positive real solutions, or 0 positive real solutions (2 may be complex).

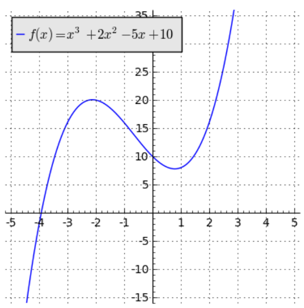
Replacing x with $-x$, we obtain $-x^3 + 2x^2 + 5x + 10 = 0$, which changes sign once between the first and second term.

Therefore, there is exactly 1 negative real solution.

The equation has at most $2 + 1 = 3$ solutions, exactly one of which is negative.

Descartes' Rule of Signs

The graph below shows that there are no positive real solutions and, as required, exactly one negative real solution.



Descartes' Rule of Signs

Example

Determine the maximum possible number of real solutions to the equation $2x^4 - 5x^3 + 3x^2 - 15x - 9 = 0$.

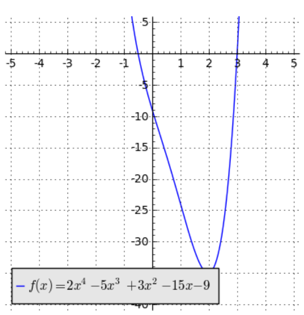
The equation changes sign 3 times, so the number of positive real solutions is either 3 or 1.

Replacing x with $-x$ yields $2x^4 + 5x^3 + 3x^2 + 15x - 9 = 0$, which changes sign once, so there must be exactly one negative real solution.

The equation has at most $3 + 1 = 4$ solutions, exactly one of which is positive and one of which is negative.

Descartes' Rule of Signs

The graph below shows that there is one positive real solution and, as required, exactly one negative real solution.



Descartes' Rule of Signs

Descartes' Rule of Signs does not identify the real solutions themselves – for that, we still need to use the IZT or RZT.

Using Descartes' Rule of Signs, however, might help us cut down the amount of checking we need to do.

For instance, if we knew that a polynomial equation had at most one positive real solution, then we could stop testing positive values with the IZT/RZT once we found that solution.

Descartes' Rule of Signs

Example

Determine any real solutions to the equation $6x^4 + 17x^3 - 22x^2 - 63x - 18 = 0$.

Using the RZT, possible real solutions are ±1, ±2, ±3, ±6, ±9, ±18, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{9}{2}$, $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, and $\pm\frac{1}{6}$.

Since the equation changes sign only once between the second and third terms, there must be exactly one positive real solution.

Testing values, $6(2)^4 + 17(2)^3 - 22(2)^2 - 63(2) - 18 = 0$, so this positive solution is at $x = 2$.

Since we have identified the positive real solution, it is no longer necessary to test any further positive values.

POLYNOMIAL EQUATIONS & INEQUALITIES

Descartes' Rule of Signs

Use synthetic division with $x = 2$.

	6	17	−22	−63	−18
2		12	58	72	18
	6	29	36	9	0

Use synthetic division again with $x = -3$.

	6	29	36	9
−3		−18	−33	−9
	6	11	3	0

The final quadratic factors as $(2x + 3)(3x + 1)$. Therefore, the quartic equation has real solutions of $x = 2$, $x = -3$, $x = -\frac{3}{2}$ and $x = -\frac{1}{3}$.

POLYNOMIAL EQUATIONS & INEQUALITIES

Questions?