MHF4U: Advanced Functions

Cofunction and "Shifting" Identities

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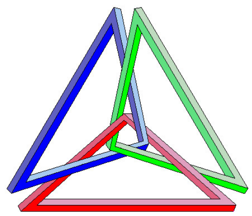
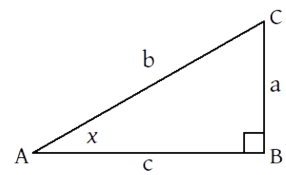


Cofunction Identities

Consider the following triangle.



Since there are π radians in a triangle, the measure of $\angle C = \pi - \frac{\pi}{2} - x = \frac{\pi}{2} - x$.

Cofunction Identities

Using the triangle, we can establish the following ratios:

$\sin x = \frac{a}{b}$	$\cos x = \frac{c}{b}$	$\tan x = \frac{a}{c}$
$\csc x = \frac{b}{a}$	$\sec x = \frac{b}{c}$	$\cot x = \frac{c}{a}$

Similarly, we can establish these additional ratios:

$\sin C = \frac{c}{b}$	$\cos C = \frac{a}{b}$	$\tan C = \frac{c}{a}$
$\csc C = \frac{b}{c}$	$\sec C = \frac{b}{a}$	$\cot C = \frac{a}{c}$

Note that $\sin x = \cos C = \frac{a}{b}$, and so $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.

Cofunction Identities

The previous identities is called a *cofunction identity*.

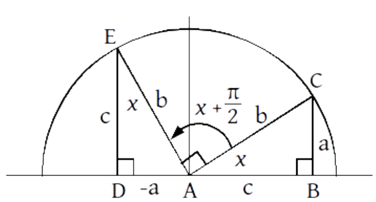
Using the same reasoning as before, the six cofunction identities are:

$\sin x = \cos\left(\frac{\pi}{2} - x\right)$	$\cos x = \sin\left(\frac{\pi}{2} - x\right)$
$\tan x = \cot\left(\frac{\pi}{2} - x\right)$	$\csc x = \sec\left(\frac{\pi}{2} - x\right)$
$\sec x = \csc\left(\frac{\pi}{2} - x\right)$	$\cot x = \tan\left(\frac{\pi}{2} - x\right)$

Cofunction identities are generally used either to simplify expressions, or to determine an equivalent expression that can be used in place of another.

Identities From Shifting

Consider the following unit circle, where $\angle BAE = x + \frac{\pi}{2}$.



Since $\angle ABC = \angle EDA$ and $\angle ACB = \angle EAD$, $\Delta ABC \cong \Delta ADE$. Therefore, $|AB| = |ED|$ and $|BC| = |DA|$.

Identities From Shifting

In ΔABC, we have the same ratios as before.

$\sin x = \frac{a}{b}$	$\cos x = \frac{c}{b}$	$\tan x = \frac{a}{c}$
$\csc x = \frac{b}{a}$	$\sec x = \frac{b}{c}$	$\cot x = \frac{c}{a}$

In ΔADE, we have the following ratios, using $\angle EAD$ as the reference angle for ΔBAE:

$\sin BAE = \frac{c}{b}$	$\cos BAE = -\frac{a}{b}$	$\tan BAE = -\frac{c}{a}$
$\csc BAE = \frac{b}{c}$	$\sec BAE = -\frac{b}{a}$	$\cot BAE = -\frac{a}{c}$

Thus, $\sin BAE = \cos x = \frac{c}{b}$, so $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

Identities From Shifting

By "shifting" each function by $\frac{\pi}{2}$, we obtain the following six identities:

$\sin\left(x+\frac{\pi}{2}\right) = \cos x$	$\cos\left(x+\frac{\pi}{2}\right) = -\sin x$
$\tan\left(x+\frac{\pi}{2}\right) = -\cot x$	$\csc\left(x+\frac{\pi}{2}\right) = \sec x$
$\sec\left(x+\frac{\pi}{2}\right) = -\csc x$	$\cot\left(x+\frac{\pi}{2}\right) = -\tan x$

Like the cofunction identities, these six identities can be used to simplify expressions, or to determine equivalent expressions.

Evaluating Trigonometric Ratios

Example

If $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$, determine the values of $\cos\frac{7\pi}{12}$ and $\cos\frac{5\pi}{12}$.

Try to express $\frac{7\pi}{12}$ as the sum or difference of $\frac{\pi}{12}$ and $\frac{\pi}{2}$.

$$\begin{aligned} x+\frac{\pi}{2} &= \frac{7\pi}{12} \\ x &= \frac{7\pi}{12}-\frac{\pi}{2} \\ &= \frac{7\pi}{12}-\frac{6\pi}{12} \\ &= \frac{\pi}{12} \end{aligned}$$

Therefore, $\frac{7\pi}{12} = \frac{\pi}{12}+\frac{\pi}{2}$.

Evaluating Trigonometric Ratios

Express $\cos\frac{7\pi}{12}$ as the sum, and use a shifting identity.

$$\begin{aligned} \cos\frac{7\pi}{12} &= \cos\left(\frac{\pi}{12}+\frac{\pi}{2}\right) \\ &= -\sin\frac{\pi}{12} \\ &= \frac{\sqrt{2}-\sqrt{6}}{4} \end{aligned}$$

Evaluating Trigonometric Ratios

Try to express $\frac{5\pi}{12}$ as the sum or difference of $\frac{\pi}{12}$ and $\frac{\pi}{2}$.

$$\begin{aligned} \frac{\pi}{2}-x &= \frac{5\pi}{12} \\ x &= \frac{\pi}{2}-\frac{5\pi}{12} \\ &= \frac{6\pi}{12}-\frac{5\pi}{12} \\ &= \frac{\pi}{12} \end{aligned}$$

Therefore, $\frac{5\pi}{12} = \frac{\pi}{2}-\frac{\pi}{12}$.

Evaluating Trigonometric Ratios

Express $\cos\frac{5\pi}{12}$ as the difference, and use a cofunction identity.

$$\begin{aligned} \cos\frac{5\pi}{12} &= \cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right) \\ &= \sin\frac{\pi}{12} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$

Evaluating Trigonometric Ratios

Example

If $\csc\frac{4\pi}{15} \approx 1.3456$, determine the values of $\sec\frac{7\pi}{30}$ and $\sec\frac{23\pi}{30}$.

Try to express $\frac{7\pi}{30}$ as the sum or difference of $\frac{4\pi}{15}$ and $\frac{\pi}{2}$.

$$\begin{aligned} \frac{\pi}{2}-x &= \frac{7\pi}{30} \\ x &= \frac{\pi}{2}-\frac{7\pi}{30} \\ &= \frac{15\pi}{30}-\frac{7\pi}{30} \\ &= \frac{4\pi}{15} \end{aligned}$$

Therefore, $\frac{7\pi}{30} = \frac{\pi}{2}-\frac{4\pi}{15}$.

Evaluating Trigonometric Ratios

Express $\sec \frac{7\pi}{30}$ as the difference, and use a cofunction identity.

$$\begin{aligned}\sec \tfrac{7\pi}{30} &= \sec\left(\tfrac{\pi}{2} - \tfrac{4\pi}{15}\right)\\ &= \csc \tfrac{4\pi}{15}\\ &\approx 1.3456\end{aligned}$$

Evaluating Trigonometric Ratios

Try to express $\frac{23\pi}{30}$ as the sum or difference of $\frac{\pi}{12}$ and $\frac{\pi}{2}$.

$$\begin{aligned}x + \tfrac{\pi}{2} &= \tfrac{23\pi}{30}\\ x &= \tfrac{23\pi}{30} - \tfrac{\pi}{2}\\ &= \tfrac{23\pi}{30} - \tfrac{15\pi}{30}\\ &= \tfrac{4\pi}{15}\end{aligned}$$

Therefore, $\frac{23\pi}{30} = \frac{4\pi}{15} + \frac{\pi}{2}$.

Evaluating Trigonometric Ratios

Express $\sec \frac{23\pi}{30}$ as the sum, and use a shifting identity.

$$\begin{aligned}\sec \tfrac{23\pi}{15} &= \sec\left(\tfrac{4\pi}{15} + \tfrac{\pi}{2}\right)\\ &= -\csc \tfrac{4\pi}{15}\\ &\approx -1.3456\end{aligned}$$

Questions?