MHF4U: Advanced Functions

Review of Basic Trigonometric Identities

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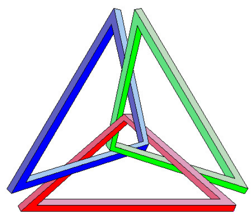


Reciprocal Identities

Recall that the three *primary* trigonometric ratios are $\sin\theta$, $\cos\theta$ and $\tan\theta$.

The three *secondary* ratios are $\csc\theta$, $\sec\theta$ and $\cot\theta$.

Since the secondary ratios are reciprocals of the primary ratios, they are often called the *reciprocal* ratios.

This relationship gives us three *trigonometric identities*.

Reciprocal Identities

For any value of θ, the reciprocal identities are $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$.

These identities can be used to prove that one trigonometric expression is equivalent to another.

Proofs Using Reciprocal Identities

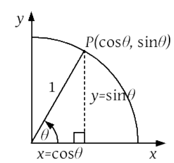
Example

Prove that $\sin\theta \cdot \csc\theta = 1$.

$$\begin{aligned} \text{LHS} &= \sin\theta \cdot \csc\theta \\ &= \sin\theta \cdot \frac{1}{\sin\theta} \\ &= \frac{\sin\theta}{\sin\theta} \\ &= 1 \\ &= \text{RHS} \end{aligned}$$

Pythagorean Identity

Let P be any point on a unit circle.



By the Pythagorean Theorem, $x^2 + y^2 = 1$.

Since $x = \cos\theta$ and $y = \sin\theta$, $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$.

Pythagorean Identity

For any value of θ, $\sin^2\theta + \cos^2\theta = 1$.

Proofs Using the Pythagorean Identity

Example

Prove that $\sin^2\theta + 4\cos^2\theta = -3\sin^2\theta + 4$.

$$\begin{aligned} \text{LHS} &= \sin^2\theta + 4\cos^2\theta \\ &= \sin^2\theta + 4(1 - \sin^2\theta) \\ &= \sin^2\theta + 4 - 4\sin^2\theta \\ &= -3\sin^2\theta + 4 \\ &= \text{RHS} \end{aligned}$$

Proofs Using the Pythagorean Identity

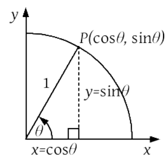
Example

Prove that $\dfrac{1 + \sin^2\theta - \cos^2\theta}{2\sin\theta} = \sin\theta$.

$$\begin{aligned} \text{LHS} &= \frac{1 + \sin^2\theta - \cos^2\theta}{2\sin\theta} \\ &= \frac{\sin^2\theta + \sin^2\theta}{2\sin\theta} \\ &= \frac{2\sin^2\theta}{2\sin\theta} \\ &= \sin\theta \\ &= \text{RHS} \end{aligned}$$

Tangent and Cotangent Identities

Let P be any point on a unit circle.



Since $\tan\theta = \frac{y}{x}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
Since $\cot\theta = \frac{1}{\tan\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

Tangent and Cotangent Identities

For any value of θ, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Proofs Using Tangent and Cotangent Identities

Example

Prove that $\sin\theta \cdot \cot\theta = \cos\theta$.

$$\begin{aligned} \text{LHS} &= \sin\theta \cdot \cot\theta \\ &= \sin\theta \cdot \frac{\cos\theta}{\sin\theta} \\ &= \cos\theta \\ &= \text{RHS} \end{aligned}$$

General Rules For Proving Identities

While there is no specific procedure for proving a trigonometric identity, the following general rules may help.

- Try to simplify, rather than expand, when possible.
- Replace all instances of $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$ with $\sin\theta$ and $\cos\theta$.
- DO NOT "move" terms or factors across an $=$ sign, since this presupposes that the identity is true.
- If necessary, work on both sides of an identity simultaneously and meet somewhere in the middle.

Proofs Using the Basic Identities

Example

Prove that $\cos\theta \cdot \cot\theta + \csc\theta = \dfrac{\cos^2\theta + 1}{\sin\theta}$.

$$\begin{aligned} \text{LHS} &= \cos\theta \cdot \cot\theta + \csc\theta \\ &= \cos\theta \cdot \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} \\ &= \frac{\cos^2\theta}{\sin\theta} + \frac{1}{\sin\theta} \\ &= \frac{\cos^2\theta + 1}{\sin\theta} \\ &= \text{RHS} \end{aligned}$$

Proofs Using the Basic Identities

Example

Prove that $\sec^2\theta - 1 = \tan^2\theta$.

$$\begin{aligned} \text{LHS} &= \sec^2\theta - 1 \\ &= \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} \\ &= \frac{1 - \cos^2\theta}{\cos^2\theta} \\ &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \tan^2\theta \\ &= \text{RHS} \end{aligned}$$

Using the Conjugate

Recall that $(x - y)(x + y) = x^2 - y^2$, a difference of squares.

We say that $x - y$ is the *conjugate* of $x + y$, and vice versa.

Similarly, $(1 - \sin\theta)(1 + \sin\theta) = 1 - \sin^2\theta$ (also a difference of squares), which simplifies to $\cos^2\theta$.

When trying to simplify rational expressions involving $1 \pm \sin\theta$ or $1 \pm \cos\theta$, it is often useful to use the conjugate and apply the Pythagorean Identity.

Using the Conjugate

Example

Prove that $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$.

$$\begin{aligned}
\text{LHS} &= \frac{\cos\theta}{1+\sin\theta} \\
&= \frac{\cos\theta}{1+\sin\theta}\cdot\frac{1-\sin\theta}{1-\sin\theta} \\
&= \frac{(\cos\theta)(1-\sin\theta)}{1-\sin^2\theta} \\
&= \frac{(\cos\theta)(1-\sin\theta)}{\cos^2\theta} \\
&= \frac{1-\sin\theta}{\cos\theta} \\
&= \text{RHS}
\end{aligned}$$

Factoring Trigonometric Expressions

Another useful technique is to factor trigonometric identities, similar to how other binomials and trinomials are factored.

If a rational expression contains the same factor in both its numerator and its denominator, then that factor can be cancelled out, leaving an equivalent expression.

All of the standard factoring techniques (common, simple, complex, perfect squares, differences of squares) may apply, and more than one method of factoring may be required.

Factoring Trigonometric Expressions

Example

Prove that $\dfrac{\cos^3\theta+\cos^2\theta}{1-\sin^2\theta} = \cos\theta + 1$.

$$\begin{aligned}
\text{LHS} &= \frac{\cos^3\theta+\cos^2\theta}{1-\sin^2\theta} \\
&= \frac{(\cos^2\theta)(\cos\theta+1)}{1-\sin^2\theta} \\
&= \frac{(\cos^2\theta)(\cos\theta+1)}{\cos^2\theta} \\
&= \cos\theta + 1 \\
&= \text{RHS}
\end{aligned}$$

Factoring Trigonometric Expressions

Example

Prove that $\dfrac{\sin^2\theta-\cos^2\theta}{\cos\theta-\sin\theta} = -\sin\theta - \cos\theta$.

$$\begin{aligned}
\text{LHS} &= \frac{\sin^2\theta-\cos^2\theta}{\cos\theta-\sin\theta} \\
&= \frac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{\cos\theta-\sin\theta} \\
&= -\frac{(\sin\theta+\cos\theta)(\cos\theta-\sin\theta)}{\cos\theta-\sin\theta} \\
&= -(\sin\theta+\cos\theta) \\
&= -\sin\theta-\cos\theta \\
&= \text{RHS}
\end{aligned}$$

Questions?