MHF4U: Advanced Functions

Addition and Subtraction Identities

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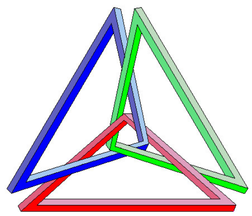
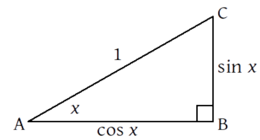


Addition Identity For Sine

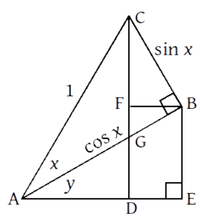
Consider the following triangle.



Since $|AC| = 1$, $|AB| = |\cos x|$ and $|BC| = |\sin x|$.

Addition Identity For Sine

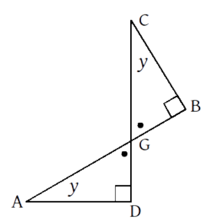
Now consider the same triangle, rotated by an angle of y.



We wish to determine a formula for $\sin(x + y)$.

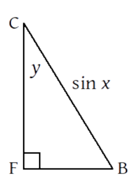
Addition Identity For Sine

Note that $\angle BCG = y$, since $\angle AGD = \angle BGC$ (opposite angles) and $\angle ADG = \angle CBG$ (both 90°).



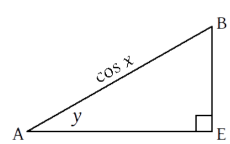
Addition Identity For Sine

In ΔCFB, $|\cos y| = \frac{|CF|}{|BC|} = \frac{|CF|}{|\sin x|}$, so $|CF| = |\sin x| \cdot |\cos y|$.



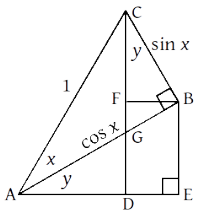
Addition Identity For Sine

In ΔABE, $|\sin y| = \frac{|BE|}{|AB|} = \frac{|BE|}{|\cos x|}$, so $|BE| = |\sin y| \cdot |\cos x|$.



Addition Identity For Sine

In ΔACD, $|AC| = 1$, so $|\sin(x+y)| = |CD|$.



$|CD| = |CF + FD| = |CF + BE|$, so
$|\sin(x+y)| = |CF + BE|$.

Addition Identity For Sine

Putting things together,
$|\sin(x+y)| = |\sin x| \cdot |\cos y| + |\sin y| \cdot |\cos x|$.

This gives us the desired formula for $\sin(x+y)$.

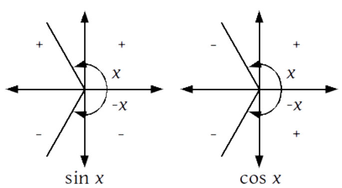
Addition Identity For Sine

For any angles x and y, $\sin(x+y) = \sin x \cos y + \sin y \cos x$.

Subtraction Identity For Sine

We can use the previous expression to derive an identity for $\sin(x-y)$.

First, recall that $\sin(-x) = -\sin x$, and $\cos(-x) = \cos x$.



Subtraction Identity For Sine

For $\sin(x-y)$, substitute $-y$ for y.

$$\begin{aligned} \sin(x+(-y)) &= \sin x \cos(-y) + \sin(-y)\cos x \\ \sin(x-y) &= \sin x \cos y - \sin y \cos x \end{aligned}$$

Subtraction Identity For Sine

For any angles x and y, $\sin(x-y) = \sin x \cos y - \sin y \cos x$.

Verifying the Sine Identities

Example

Verify that $\sin\left(\frac{5\pi}{6} - \frac{2\pi}{3}\right) = \sin\frac{5\pi}{6}\cos\frac{2\pi}{3} - \sin\frac{2\pi}{3}\cos\frac{5\pi}{6}$.

$$\begin{aligned} \sin\left(\tfrac{5\pi}{6} - \tfrac{2\pi}{3}\right) &= \sin\left(\tfrac{5\pi}{6} - \tfrac{4\pi}{6}\right) \\ &= \sin\tfrac{\pi}{6} \\ &= \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} \sin\tfrac{5\pi}{6}\cos\tfrac{2\pi}{3} - \sin\tfrac{2\pi}{3}\cos\tfrac{5\pi}{6} &= \tfrac{1}{2}\left(-\tfrac{1}{2}\right) - \tfrac{\sqrt{3}}{2}\left(-\tfrac{\sqrt{3}}{2}\right) \\ &= -\tfrac{1}{4} + \tfrac{3}{4} \\ &= \tfrac{1}{2} \end{aligned}$$

Addition Identity For Cosine

To develop an addition formula for cosine, we can make use of the cofunction identities developed earlier.

Since $\cos(a) = \sin\left(\frac{\pi}{2} - a\right)$, let $a = x + y$.

$$\begin{aligned} \cos(x+y) &= \sin\left(\tfrac{\pi}{2} - [x+y]\right) \\ &= \sin\left(\left[\tfrac{\pi}{2} - x\right] - y\right) \end{aligned}$$

Use the subtraction formula for sine next.

$$\begin{aligned} \cos(x+y) &= \sin\left(\tfrac{\pi}{2} - x\right)\cos y - \sin y \cos\left(\tfrac{\pi}{2} - x\right) \\ &= \cos x \cos y - \sin y \sin x \end{aligned}$$

Addition Identity For Cosine

For any angles x and y, $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Subtraction Identity For Cosine

A similar formula can be developed for subtraction using cosine.

$$\begin{aligned}\cos(x-y) &= \cos(x+(-y))\\ &= \cos x\cos(-y) - \sin x\sin(-y)\\ &= \cos x\cos y + \sin x\sin y\end{aligned}$$

Subtraction Identity For Cosine

For any angles x and y, $\cos(x-y) = \cos x\cos y + \sin x\sin y$.

Verifying the Cosine Identities

Example

Verify that $\cos\left(\frac{\pi}{3}+\pi\right) = \cos\frac{\pi}{3}\cos\pi - \sin\frac{\pi}{3}\sin\pi$.

$$\begin{aligned}\cos\left(\tfrac{\pi}{3}+\pi\right) &= \cos\left(\tfrac{\pi}{3}+\tfrac{3\pi}{3}\right)\\ &= \cos\tfrac{4\pi}{3}\\ &= -\tfrac{1}{2}\end{aligned}$$

$$\begin{aligned}\cos\tfrac{\pi}{3}\cos\pi - \sin\tfrac{\pi}{3}\sin\pi &= \tfrac{1}{2}(-1) - \tfrac{\sqrt{3}}{2}(0)\\ &= -\tfrac{1}{2}\end{aligned}$$

Exact Value of a Trigonometric Ratio

Example

Determine an exact value for $\cos\frac{\pi}{12}$.

Begin by expressing $\frac{\pi}{12}$ as the sum or difference of two angles for which we know their exact values.

$$\begin{aligned}\tfrac{\pi}{12} &= \tfrac{4\pi}{12} - \tfrac{3\pi}{12}\\ &= \tfrac{\pi}{3} - \tfrac{\pi}{4}\end{aligned}$$

Now, use the subtraction formula for cosine.

$$\begin{aligned}\cos\tfrac{\pi}{12} &= \cos\left(\tfrac{\pi}{3}-\tfrac{\pi}{4}\right)\\ &= \cos\tfrac{\pi}{3}\cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{3}\sin\tfrac{\pi}{4}\\ &= \tfrac{1}{2}\cdot\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2}\\ &= \tfrac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

Questions?