MDM4U: Mathematics of Data Management

To Count Or Not To Count

The Principle of Inclusion and Exclusion

J. Garvin

Principle of Inclusion and Exclusion

In how many ways can you draw a black face card or a spade from a standard deck of playing cards?

There are 6 black face cards in the deck, and 13 spades. This gives a total of $6 + 13 = 19$ ways.

But. . .

Listing all of the possible spades and black face cards shows that there are only 16 ways: A♠, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠, J♣, Q♣, K♣

We overcounted the 3 cards that are both spades *and* black face cards. Note that $6 + 13 - 3 = 16$.

Principle of Inclusion and Exclusion

Principle of Inclusion and Exclusion for Two Sets

For any two non-disjoint sets A and B, the number of elements in either A or B is given by $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Explanation: By adding $n(A)$ and $n(B)$, we have overcounted the elements in the overlapping area $A \cap B$ by adding them in twice.

To remedy this, subtract the intersection $A \cap B$. This describes the formula above.

Principle of Inclusion and Exclusion

Example

How many positive integers, less than or equal to 100, are divisible by two or five?

Let T be the set of numbers divisible by two, and F the set of numbers divisible by five. Then:

- $n(T) = 50$
- $n(F) = 20$
- $n(T \cap F) = 10$

Therefore, $n(T) + n(F) - n(T \cap F) = 50 + 20 - 10 = 60$ numbers are divisible by two or five.

Principle of Inclusion and Exclusion

Example

In a survey, 20 students said that they enjoyed volleyball, while 15 enjoyed basketball, and 8 said they enjoyed both sports. How many students completed the survey?

If V is the set of students who enjoy volleyball, and B is the set of students who enjoy basketball, then:

- $n(V) = 20$
- $n(B) = 15$
- $n(V \cap B) = 8$

Therefore, $n(V) + n(B) - n(V \cap B) = 20 + 15 - 8 = 27$ students completed the survey.

Principle of Inclusion and Exclusion

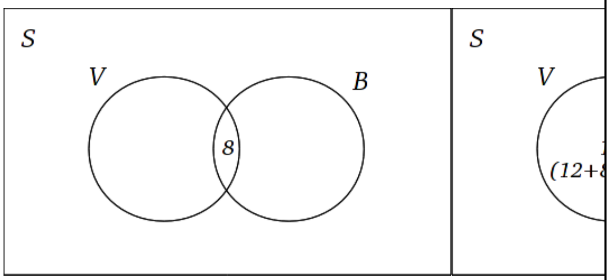
Another way to visualize scenarios like this involving two non-disjoint sets is by using Venn diagrams.

Beginning with the intersection, work outwards to fill in the number of students.

Since the intersection includes some elements from each set, subtract its value from that of each set.

When the diagram is completed, add up all of the values to determine the count.

Principle of Inclusion and Exclusion



Principle of Inclusion and Exclusion

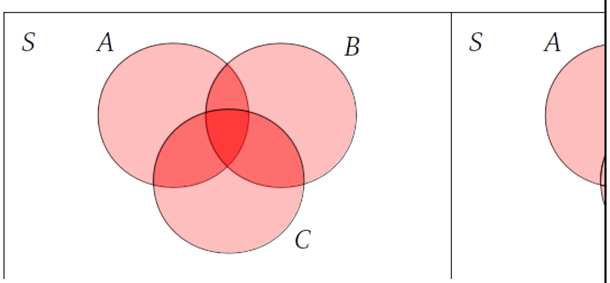
Principle of Inclusion and Exclusion for Three Sets

For any three non-disjoint sets A, B and C, the number of elements in either A, B or C is given by $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

Explanation: By adding $n(A)$, $n(B)$ and $n(C)$, we have overcouted the elements in the overlapping areas $A \cap B$, $A \cap C$ and $B \cap C$ and, by extension, have included the area $A \cap B \cap C$ three times.

Subtracting each of these three areas completely removes the elements in $A \cap B \cap C$, so it must be added back in again. This describes the formula above.

Principle of Inclusion and Exclusion



Principle of Inclusion and Exclusion

Example

Of 100 dogs in a dog show, 44 have blue eyes, 45 have white fur, 42 have docked ears, 20 have blue eyes and white fur, 16 have white fur and docked ears, 14 have docked ears and blue eyes, and 9 have all three traits. How many dogs at the show do not have any of these traits?

The Principle of Inclusion and Exclusion can be used to find the number of dogs that possess any or all of these characteristics.

Since we are asked to find the number of dogs *without* these characteristics, we can subtract our answer from 100 (the number of dogs at the show).

Principle of Inclusion and Exclusion

If B is the set of dogs with blue eyes, W the set of dogs with white fur, and D the set of dogs with docked ears, then:

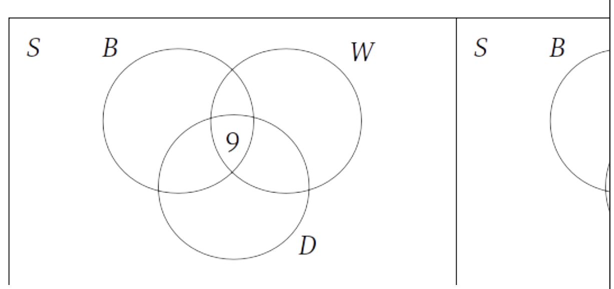
- $n(B) = 44$
- $n(W) = 45$
- $n(D) = 42$
- $n(B \cap W) = 20$
- $n(W \cap D) = 16$
- $n(B \cap D) = 14$
- $n(B \cap W \cap D) = 9$

Thus, there are $n(B) + n(W) + n(D) - n(B \cap W) - n(W \cap D) - n(B \cap D) + n(B \cap W \cap D) = 44 + 45 + 42 - 20 - 16 - 14 + 9 = 90$ dogs that have at least one of these characteristics.

Therefore, $100 - 90 = 10$ dogs have none of these traits.

Principle of Inclusion and Exclusion

A Venn diagram could have been used as well.



The total of the areas is 90, and $100 - 90 = 10$.

Questions?