MDM4U: Mathematics of Data Management

Arranging Identical Items

Permutations with Repetition

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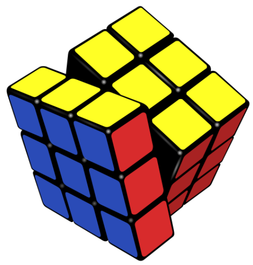


Recap

Example

In how many ways can the letters of the word MATH be arranged?

There are ${}_4P_4 = 4! = 24$ permutations of the four letters.

This can be verified by enumerating all of the possibilities:

MATH	MAHT	MTAH	MTHA	MHAT	MHTA
AMTH	AMHT	AHMT	AHTM	ATHM	ATMH
TAMH	TAHM	TMAH	TMHA	THAM	THMA
HAMT	HATM	HMAT	HMTA	HTAM	HTMA

Recap

Example

In how many ways can the letters of the word DATA be arranged?

There are ${}_4P_4 = 4! = 24$ permutations of four items.

This can be verified by enumerating all of the possibilities:

DATA	DAAT	DTAA	TADA	TAAD	TDAA
AADT	AATD	ADAT	ADTA	ATDA	ATAD

Wait, what?

Permutations with Repetition

In the case of MATH, all of the letters were distinct. With DATA, however, the letter A occurred twice.

Although there are still $4! = 24$ permutations, some of them are indistinguishable.

To see this more clearly, colour one A red and the other blue:

DATA	DATA	DAAT	DAAT	DTAA	DTAA
TADA	TADA	TAAD	TAAD	TDAA	TDAA
ATAD	ATAD	AADT	AADT	AATD	AATD
ADAT	ADAT	ATDA	ATDA	ADTA	ADTA

So, we are overcounting when using the previous permutation formula.

Permutations with Repetition

Permutations with Some Identical Items

Given n items, with a identical items of one type, b identical items of another, c identical items of another, and so forth, the number of permutations of all n items is $\frac{n!}{a!b!c!\ldots}$

Proof: Colour each of the a identical items of the first type different colours. There are $a!$ ways of arranging these items, each of which would produce the same result.

The same applies to the b identical items of the second type, the c identical items of the third type. In each case, we are overcounting by $b!$, then by $c!$.

To remedy this, we must divide the total number of permutations, $n!$, by $a!$, then $b!$, then $c!$, etc.

Permutations with Repetition

Check

In how many ways can the letters of the word DATA be arranged?

There are $\frac{4!}{2!} = \frac{24}{2} = 12$ ways to arrange the four letters.

Permutations with Repetition

Your Turn

In how many ways can the letters of the word MISSISSAUGA be arranged?

There are eleven letters, including four Ss, two Is and two As.

Therefore, there are $\frac{11!}{4!2!2!} = 415\,800$ ways to arrange the letters.

Permutations with Repetition

Example

Recall the earlier example about binary numbers, which use only 0 and 1 as allowable digits. How many eight-bit bytes contain exactly three ones?

There are three ones, and five zeroes.

Therefore, there are $\frac{8!}{3!5!} = 56$ bytes with exactly three ones.

Which way is easier?

Permutations with Repetition

Example

In how many ways can ten marbles (two yellow, three red and five blue) be arranged in a line if the two yellow marbles must be on the ends?

The two yellow marbles are fixed, so there are really only eight items to arrange.

This can be done in $\frac{8!}{3!5!} = 56$ ways.

This is the same as the previous example!

Permutations with Repetition

Your Turn

In how many ways can the letters of the word PARALLEL be arranged if the two As cannot be beside each other?

There are eight letters, including two As and three Ls. These can be arranged in $\frac{8!}{2!3!} = 3\,360$ ways.

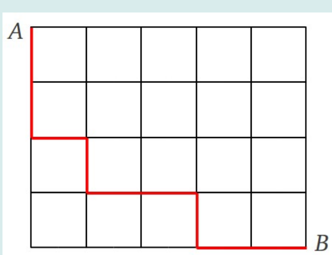
If the As *are* together ("AA"), there are seven items to arrange, including three Ls. These can be arranged in $\frac{7!}{3!} = 840$ ways.

Using an indirect method, the number of ways to arrange the letters such that the two As are not together is $\frac{8!}{2!3!} - \frac{7!}{3!} = 2\,520$.

Permutations with Repetition

Example

How many shortest paths are there from Point A to Point B? An example is shown in red.



Permutations with Repetition

A shortest path would move directly from A toward B, by moving either East or South.

Let E denote a movement to the East, and S a movement to the South.

Some possible paths, then, include:

- EEEEESSSS
- ESESESESE
- SSESEESEE

A path, then, is simply an arrangement of five Es and four Ss.

Therefore, there are $\frac{9!}{5!4!} = 126$ shortest paths.

Questions?