MDM4U: Mathematics of Data Management

Applying Counting Fundamentals

Counting Principles, Part 2

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More Counting Principles

A palindrome is a word, phrase, number or other sequence of units that can be read the same way forward or backward.

radar

Was it a rat I saw?

A man, a plan, a canal: Panama

123 321

More Counting Principles

Example

How many six-digit integers are palindromes?

A six-digit integer cannot begin with a leading zero, so there are nine choices (1-9) for the first digit.

Once the first digit is chosen, the last digit is fixed. It *must* be the same as the first digit, so there is only one choice.

The second digit can be any number, so there are ten options. The second-to-last digit must be the same, so there is only one choice.

The third digit can be any number, so there are ten options. The third-last digit has only one choice.

According to the FCP, there are $9 \times 10 \times 10 \times 1 \times 1 \times 1 = 900$ palindromic six-digit integers.

More Counting Principles

Example

How many six-digit integers are palindromes, if a specific number appears only twice?

There are nine choices for the first digit (1-9). The last digit is fixed.

There are also nine choices for the second digit (0-9, minus the previously chosen digit). The second-to-last digit matches this.

There are eight choices for the third digit (0-9, minus the previous two). The third-last digit matches this.

According to the FCP, there are $9 \times 9 \times 8 \times 1 \times 1 \times 1 = 648$ palindromic six-digit integers with distinct pairs of digits.

More Counting Principles

Example

How many six-digit integers are palindromes that are odd?

Solution: There are five choices for the last digit (1, 3, 5, 7, and 9). The first digit is fixed.

There are ten choices for the second digit, which matches the second-to-last.

There are ten choices for the third digit, which matches the third-last.

According to the FCP, there are $1 \times 10 \times 10 \times 1 \times 1 \times 5 = 500$ palindromic six-digit integers with distinct digits.

Overcounting

Computers use the binary number system, which uses only the digits 0 and 1, to represent data.

For example, the numbers 1 through 4 in binary are 1, 10, 11 and 100.

Each digit is called a "bit." Eight bits make a "byte."

For example, $45_{10} = 00101101_2$.

Overcounting

Example

How many eight-bit bytes are possible?

Solution: There are...

- ...two options (0 or 1) for the first digit
- ...two options (0 or 1) for the second digit
- ...two options (0 or 1) for the third digit
- ...
- ...two options (0 or 1) for the eighth digit

According to the FCP, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$ possibilities.

Overcounting

Example

How many eight-bit bytes contain exactly three ones?

Solution: There are...

- ...eight available positions for the first 1
- ...seven available positions for the second 1
- ...six available positions for the third 1

All of the remaining bits must be zeroes. According to the FCP, there are $8 \times 7 \times 6 = 336$ possibilities.

Huh?

Overcounting

The solution can't make sense, since there are only 256 possible eight-bit bytes. What happened?

Consider the case where we select position one for the first 1, position two for the second 1, and position three for the third 1. This gives the byte 11100000.

Now consider the case where we select position three for the first 1, position two for the second 1, and position one for the third 1. This gives the byte 11100000.

These are the same byte! We've overcounted!

Overcounting

To correct our calculation, we must get rid of all of the situations that have been overcounted.

There are six ways to arrange the three 1s.

111, 111, 111, 111, 111, 111

By choosing positions, we have overcounted by a factor of six, and must divide our answer accordingly.

Therefore, the total number of eight-bit bytes with exactly three ones is $336 \div 6 = 56$.

We will discover a much easier way to handle situations like this later.

Overcounting

Your Turn

Recall that a Canadian postal code has the format A9A 9A9. How many possible postal codes are there that contain exactly two Zs?

There are three positions for the first Z, and two positions for the second Z. The remaining letter cannot be a Z, so there are 25 choices.

Therefore, there are $3 \times 2 \times 25 = 150$ ways to fill in the two Zs and additional letter.

Overcounting

Next, we can choose any digit for each of the three numbers. Thus, there are $10 \times 10 \times 10 = 1\,000$ possibilities for the numbers.

By the FCP, there are $150 \times 1\,000 = 150\,000$ ways to choose the letters, then the numbers.

However, since the two Zs can be arranged in 2 ways, we must divide this by 2. This gives $150\,000 \div 2 = 75\,000$ postal codes.

Overcounting

An alternate solution, based on cases, is as follows.

Of the three positions for letters, the two Zs can be placed in three ways (first/second, first/third, or second/third). The third letter can be any of the remaining 25 letters. This gives us $3 \times 25 = 75$ possibilities for the three letters.

There are 10 possibilities for each of the three numbers, for a total of $10 \times 10 \times 10 = 1\,000$ possibilities.

Therefore, the total number of postal codes with exactly two Zs is $75 \times 1\,000 = 75\,000$.

Which method is easier?

Indirect Method of Counting

Sometimes, when trying to count all possibilities for a given situation, we consider different cases.

In some instances, however, there are too many cases to consider.

In these instances, it may be more efficient to use an *indirect method* of counting instead.

By calculating all possibilities without any restrictions, and subtracting those cases that *do not* meet a set of criteria, we are left with all cases that *do* meet those criteria.

The next example illustrates this method.

Indirect Method of Counting

Example

How many three-digit positive integers contain at least one 9?

Without restrictions, there are $9 \times 10 \times 10 = 900$ positive three-digit integers.

To count all positive three-digit integers without any 9s, remove 9 from the possibilities. This gives $8 \times 9 \times 9 = 648$ integers without any 9s.

Using an indirect method, there must be $900 - 648 = 252$ integers with at least one 9.

Indirect Method of Counting

Your Turn

Determine the number of postal codes that contain one or more Zs.

There are $26^3 \times 10^3 = 17\,576\,000$ possible postal codes without any restrictions.

By disallowing Z as an option, there are $25^3 \times 10^3 = 15\,625\,000$ postal codes that do not contain any Zs.

Therefore, using an indirect method, there are $17\,576\,000 - 15\,625\,000 = 1\,951\,000$ postal codes that contain one or more Zs.

Questions?