MDM4U: Mathematics of Data Management

Simplifying the Distributive Property

The Binomial Theorem

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Expanding Polynomials

Expand and simplify $(x+y)^2$.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= x^2+xy+xy+y^2\\ &= 1x^2+2xy+1y^2\end{aligned}$$

Hmmmm...

Expanding Polynomials

Expand and simplify $(x+y)^3$.

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(x^2+2xy+y^2)\\ &= x^3+2x^2y+xy^2+x^2y+2xy^2+y^3\\ &= 1x^3+3x^2y+3xy^2+1y^3\end{aligned}$$

Looks familiar...

Expanding Polynomials

Expand and simplify $(x+y)^4$.

$$(x+y)^4 = 1x^4+4x^3y+6x^2y^2+4xy^3+1y^4$$

Can you spot the pattern?

Expanding Polynomials

The coefficients of the simplified polynomial expression correspond to entries in Pascal's Triangle.

The coefficients in the expansion of $(x+y)^n$ are the entries in Row n.

The exponents follow a predictable pattern too.

The exponent for x decreases by 1, while the exponent for y increases by 1.

In each term, the sum of the exponents is n.

Binomial Theorem

$$(x+y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_nx^0y^n$$

Binomial Theorem

Consider the expansion of $(x+y)^5$.

$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$

The terms of the simplified polynomial are formed by selecting one, and *only* one, x or y from each of the five factors in all possible ways and multiplying them together.

For example, we can select an x from 3 of the 5 factors and a y from the remaining 2 factors. This results in $xxxyy = x^3y^2$.

The coefficient of x^3y^2 is the number of ways in which an x can be selected from the 3 factors, and a y from the remaining 2 factors. This can be done in ${}_5C_3$ ways.

Binomial Theorem

- x from 5 factors, chosen in ${}_5C_5$ ways, gives ${}_5C_5x^5$
- x from 4 factors, chosen in ${}_5C_4$ ways, gives ${}_5C_4x^4y$
- x from 3 factors, chosen in ${}_5C_3$ ways, gives ${}_5C_3x^3y^2$
- x from 2 factors, chosen in ${}_5C_2$ ways, gives ${}_5C_2x^2y^3$
- x from 1 factors, chosen in ${}_5C_1$ ways, gives ${}_5C_1xy^4$
- x from 0 factors, chosen in ${}_5C_0$ ways, gives ${}_5C_0y^5$

Therefore
$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

Binomial Theorem

Example

Expand $(3a+4b)^3$.

Solution: Use the substitutions $x = 3a$, $y = 4b$ and $n = 3$.

$$\begin{aligned}(3a+4b)^3 &= {}_3C_0(3a)^3 + {}_3C_1(3a)^2(4b) + {}_3C_2(3a)(4b)^2 + {}_3C_3(4b)^3 \\ &= 1(27a^3) + 3(9a^2)(4b) + 3(3a)(16b^2) + 1(64b^3) \\ &= 27a^3 + 108a^2b + 144ab^2 + 64b^3\end{aligned}$$

Binomial Theorem

Example

Expand $(2a+3b)^4$.

Solution: Use the substitutions $x = 2a$, $y = 3b$ and $n = 4$.

$$\begin{aligned}(2a+3b)^4 =& {}_4C_0(2a)^4 + {}_4C_1(2a)^3(3b) + {}_4C_2(2a)^2(3b)^2 \\ &+ {}_4C_3(2a)(3b)^3 + {}_4C_4(3b)^4 \\ =& 1(16a^4) + 4(8a^3)(3b) + 6(4a^2)(9b^2) \\ &+ 4(2a)(27b^3) + 1(81b^4) \\ =& 16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4\end{aligned}$$

Binomial Theorem

Notice that in the expansion of $(3a+4b)^3$ there were 4 terms, while in the expansion of $(2a+3b)^4$ there were 5.

In general, the expansion of $(x+y)^n$ will have $n+1$ terms.

The kth term will have a coefficient of ${}_nC_{k-1}$, and exponents $x^{n-k+1}y^{k-1}$.

Binomial Theorem

Example

How many terms are in the expansion of $(a+b)^{20}$, and which term contains $a^{12}b^8$?

Since $n = 20$, there are 21 terms in the expansion.

To determine the kth term containing $a^{12}b^8$, let $k-1 = 8$. Therefore, $k = 9$.

Binomial Theorem

Example

Expand $(4a-2b)^4$.

Solution: Use the substitutions $x = 4a$, $y = -2b$ and $n = 4$. Note that y is negative. What will happen?

$$\begin{aligned}(4a-2b)^4 =& {}_4C_0(4a)^4 + {}_4C_1(4a)^3(-2b) + {}_4C_2(4a)^2(-2b)^2 \\ &+ {}_4C_3(4a)(-2b)^3 + {}_4C_4(-2b)^4 \\ =& 1(256a^4) + 4(64a^3)(-2b) + 6(16a^2)(4b^2) \\ &+ 4(4a)(-8b^3) + 1(16b^4) \\ =& 256a^4 - 512a^3b + 384a^2b^2 - 128ab^3 + 16b^4\end{aligned}$$

If y is negative, the terms in the expansion alternate signs.

Questions?