MCV4U: Calculus & Vectors

Velocities as Vectors

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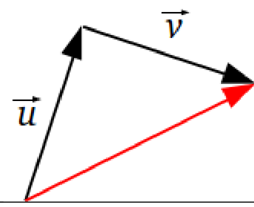


Velocity

Recall that speed, a measure of how fast something is travelling, is a scalar quantity.

Velocity is speed with direction, and is a vector quantity.

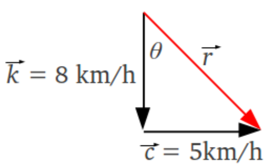
By using vectors to represent velocities, it is possible to solve a variety of problems.

Resultant Velocity Problems

Example

A kayaker paddles 8 km/h due south across a river that has a current flowing 5 km/h due east. What is the resulting velocity of the kayaker?

Use the following diagram, where $\vec{k}$ is the velocity of the kayaker, $\vec{c}$ is the velocity of the current, and $\vec{r}$ is the resultant velocity.



Resultant Velocity Problems

Use the Pythagorean Theorem to find the kayaker's speed.

$$\begin{aligned} |\vec{r}| &= \sqrt{8^2+5^2} \\ &= \sqrt{89} \\ &\approx 9.4 \text{ km/h} \end{aligned}$$

Use a trigonometric ratio to find the direction.

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{5}{8}\right) \\ &\approx 32^\circ \end{aligned}$$

Therefore, the resulting velocity is approximately 9.4 km/h S32°E.

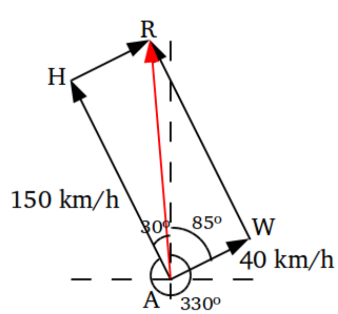
Resultant Velocity Problems

Example

An airplane is travelling on a bearing of 330° at a constant speed of 150 km/h. A wind blows on a bearing of 85° at 40 km/h. Determine the speed and direction of the airplane relative to the ground.

Use the following diagram, where $\vec{AH}$ is the airplane's velocity, $\vec{AW}$ is the wind's velocity, and $\vec{AR}$ is the resultant velocity of the airplane relative to the ground.

Resultant Velocity Problems



Resultant Velocity Problems

To determine the velocity of the airplane relative to the ground, we need to determine the magnitude of $\vec{AR}$.

From the given bearings, $\angle WAH = 30^\circ + 85^\circ = 115^\circ$, and $\angle AHR = 180^\circ - 115^\circ = 65^\circ$.

Use the cosine law to determine $|\vec{AR}|$.

$$\begin{aligned} |\vec{AR}| &= \sqrt{|\vec{HR}|^2 + |\vec{AH}|^2 - 2(|\vec{HR}|)(|\vec{AH}|)\cos(AHR)} \\ &= \sqrt{40^2 + 150^2 - 2(40)(150)\cos(65^\circ)} \\ &\approx 138 \text{ km/h} \end{aligned}$$

Resultant Velocity Problems

To determine the direction, use the sine law (or cosine law) to find the measure of $\angle HAR$, then add it to the airplane's original bearing.

$$\begin{aligned} \frac{\sin(HAR)}{|\vec{HR}|} &= \frac{\sin(AHR)}{|\vec{AR}|} \\ \frac{\sin(HAR)}{40} &\approx \frac{\sin(65^\circ)}{138} \\ \angle HAR &\approx \sin^{-1}\left(\frac{40\sin(65^\circ)}{138}\right) \\ &\approx 15^\circ \end{aligned}$$

Therefore, the actual bearing of the airplane is approximately $330^\circ + 15^\circ$, or 345°. Its actual speed is 138 km/h.

Problems Given a Resultant Velocity

Example

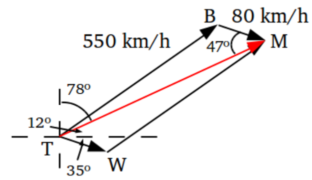
A pilot wishes to fly 508 km from Toronto to Montréal, on a bearing of 78°. The airplane has a top speed of 550 km/h. An 80 km/h wind is blowing on a bearing of 125°.

- In what direction should the pilot fly to reach the destination?
- At what speed will the plane be travelling?
- How long will the trip take?

Since we are given information about the resultant, we need to work backward!

Problems Given a Resultant Velocity

In the diagram, $\vec{TM}$ represents the resultant trip from Toronto to Montréal. Let $\vec{TB}$ be the velocity of the plane on its new bearing and let $\vec{TW}$ be the velocity of the wind.



Note that $\angle TMB = \angle MTW = 12^\circ + 35^\circ = 47^\circ$.

Problems Given a Resultant Velocity

We can use the sine law to find the measure of $\angle BTM$.

$$\begin{aligned} \frac{\sin(BTM)}{|\vec{BM}|} &= \frac{\sin(BMT)}{|\vec{TB}|} \\ \frac{\sin(BTM)}{80} &\approx \frac{\sin(47^\circ)}{550} \\ \angle BTM &\approx \sin^{-1}\left(\frac{80\sin(47^\circ)}{550}\right) \\ &\approx 6.1^\circ \end{aligned}$$

The pilot must fly at a bearing of approximately $78^\circ - 6.1^\circ \approx 71.9^\circ$.

Problems Given a Resultant Velocity

Now that we know the bearing, we can calculate the speed of the airplane using the cosine law.

$\angle TBM \approx 180^\circ - 47^\circ - 6.1^\circ \approx 126.9^\circ$.

$$\begin{aligned} |\vec{TM}| &= \sqrt{|\vec{TB}|^2 + |\vec{BM}|^2 - 2(|\vec{TB}|)(|\vec{BM}|)\cos(TBM)} \\ &\approx \sqrt{550^2 + 80^2 - 2(550)(80)\cos(126.9^\circ)} \\ &\approx 601 \text{ km/h} \end{aligned}$$

Since Montréal is 508 km away, a plane travelling at 601 km/h will make the trip in $508/601 \approx 0.85$ hours, or 51 minutes.

Relative Velocity

Relative velocity is what an observer perceives when s/he perceives her/himself to be stationary.

It is the difference of two velocities.

Relative Velocity

When two objects, A and B, have velocities $\vec{v_A}$ and $\vec{v_B}$, the velocity of B relative to A is $\vec{v_{rel}} = \vec{v_B} - \vec{v_A}$.

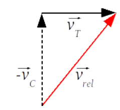
Relative Velocity

Example

A truck is travelling east at 60 km/h, while a car is travelling south at 80 km/h. What is the relative velocity of the truck to the car?

Let $\vec{v_T}$ be the velocity of the truck and $\vec{v_C}$ the velocity of the car. We want $\vec{v_T} - \vec{v_C}$, or $\vec{v_T} + (-\vec{v_C})$.

While $-\vec{v_C}$ has the opposite direction as $\vec{v_C}$, the two vectors are still at right angles to each other.



Relative Velocity

Using the Pythagorean Theorem:

$$|\vec{v_{rel}}| = \sqrt{80^2 + 60^2} = 100 \text{ km/h}$$

Using the tangent ratio for the direction:

$$\theta = \tan^{-1}\left(\frac{60}{80}\right) \approx 37^\circ$$

Therefore, the velocity of the truck relative to the car is 100 km/h, at a bearing of approximately N37°E.

Questions?