MCV4U: Calculus & Vectors

Slope of a Tangent

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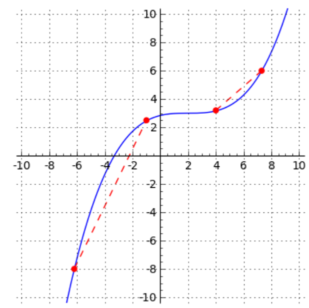
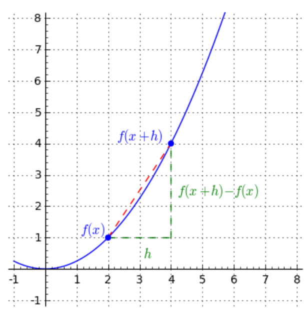
Slope of a Secant

Recall that a *secant* is a line segment that connects two points on a curve, such as the two secants below.



Slope of a Secant

The slope of the secant depends on the magnitude of the interval, h, over which it is taken.



Difference Quotient

Recall that slope is defined as "rise over run".

$$\begin{aligned} m_{\text{secant}} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(x+h)-f(x)}{(x+h)-x} \\ &= \frac{f(x+h)-f(x)}{h} \end{aligned}$$

This formula is known as the *difference quotient*.

Difference Quotient

The difference quotient, $m_{\text{secant}} = \frac{f(x+h)-f(x)}{h}$, gives the slope of the secant over the interval $[x, x+h]$.

Difference Quotient

Example

Determine the slope of the secant to $f(x) = x^3 - 5$ on the interval $[-1, 4]$.

The magnitude of the interval is $h = 4 - (-1) = 5$.

When $x = -1$, $f(-1) = (-1)^3 - 5 = -6$.

When $x = 4$, $f(4) = (4)^3 - 5 = 59$.

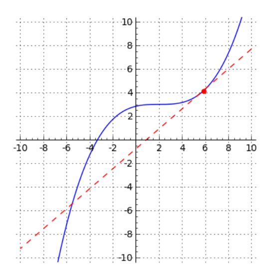
Substitute $h = 4 - (-1) = 5$, $f(-1) = -6$ and $f(4) = 59$ into the difference quotient.

$$\begin{aligned} m_{\text{secant}} &= \frac{59 - (-6)}{5} \\ &= 13 \end{aligned}$$

The slope of the secant is 13 on the interval $[-1, 4]$.

Slope of a Tangent

A *tangent* is a line that "just touches" a curve, such as the tangent at $x = 6$ below.



Note that a tangent *may* cross a function at other points.

Slope of a Tangent

As the interval, h, of the secant gets smaller, the slope of the secant better approximates the behaviour of the function at a given point P.

If h is sufficiently small, the slope of the secant will approach the slope of the tangent at P.

In previous courses, you have estimated the slope of the tangent by using a very small interval, such as $h = 0.00001$.

While this is "good enough" for many functions, a more precise method is better.

Difference Quotient (Again)

To calculate the slope of the tangent at a given point, we need to evaluate the difference quotient as $h \to 0$.

Since setting $h = 0$ will result in division by zero, it is necessary to rewrite the difference quotient so that this restriction is eliminated.

Fortunately, if all steps are done correctly, this should happen naturally.

Difference Quotient (Again)

Example

Determine the slope of the tangent to $f(x) = x^2 - 4$ at $x = 3$.

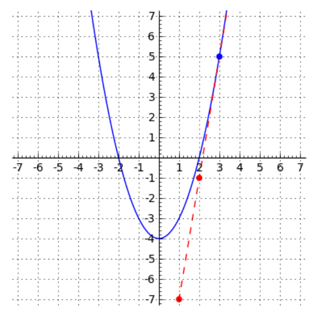
When $x = 3$, $f(3) = 3^2 - 4 = 5$, so the point of tangency is $(3, 5)$.

Substitute $x = 3$ and $f(3) = 5$ into the difference quotient.

$$\begin{aligned} m_{\text{tangent}} &= \frac{(3+h)^2 - 4 - 5}{h} \\ &= \frac{9 + 6h + h^2 - 9}{h} \\ &= \frac{h(6+h)}{h} \\ &= 6 + h \end{aligned}$$

Difference Quotient (Again)

As $h \to 0$, $6 + h \to 6$. Therefore, the slope of the tangent to $f(x) = x^2 - 4$ at $(3, 5)$ is 6.



Difference Quotient (Again)

Example

Determine the slope of the tangent to $f(x) = \sqrt{4x+1}$ at $x = 2$.

When $x = 2$, $f(2) = \sqrt{4(2)+1} = 3$, so the point of tangency is $(2, 3)$.

Substitute $x = 2$ and $f(2) = 3$ into the difference quotient.

$$m_{\text{tangent}} = \frac{\sqrt{4(2+h)+1} - 3}{h}$$

To eliminate the h in the denominator, we can multiply the numerator and denominator by the *conjugate* of the denominator.

Difference Quotient (Again)

$$\begin{aligned} m_{\text{tangent}} &= \frac{\sqrt{4(2+h)+1} - 3}{h} \times \frac{\sqrt{4(2+h)+1} + 3}{\sqrt{4(2+h)+1} + 3} \\ &= \frac{4(2+h) + 1 - 9}{h(\sqrt{4(2+h)+1} + 3)} \\ &= \frac{4h}{h(\sqrt{9+4h} + 3)} \\ &= \frac{4}{\sqrt{9+4h} + 3} \end{aligned}$$

As $h \to 0$, $\frac{4}{\sqrt{9+4h}+3} \to \frac{4}{6} = \frac{2}{3}$. Thus, the slope to $f(x) = \sqrt{4x+1}$ at $(2, 3)$ is $\frac{2}{3}$.

Difference Quotient (Again)

Example

Determine the slope of the tangent to $f(x) = \dfrac{x-4}{x}$ at $x = 5$.

When $x = 5$, $f(5) = \dfrac{5-4}{5} = \frac{1}{5}$, so the point of tangency is $\left(5, \frac{1}{5}\right)$. Substitute into the difference quotient.

$$\begin{aligned} m_{\text{tangent}} &= \frac{\frac{5+h-4}{5+h} - \frac{1}{5}}{h} \\ &= \frac{\frac{1+h}{5+h} - \frac{1}{5}}{h} \end{aligned}$$

Find a common denominator for the terms in the numerator, and simplify.

Difference Quotient (Again)

$$\begin{aligned} m_{\text{tangent}} &= \frac{\frac{1+h}{5+h} - \frac{1}{5}}{h} \\ &= \frac{\frac{5(1+h)-1(5+h)}{5(5+h)}}{h} \\ &= \frac{4h}{5h(5+h)} \\ &= \frac{4}{5(5+h)} \end{aligned}$$

As $h \to 0$, $\dfrac{4}{5(5+h)} \to \frac{4}{25}$. Thus, the slope to $f(x) = \dfrac{x-4}{x}$ at $\left(5, \frac{1}{5}\right)$ is $\frac{4}{25}$.

Questions?