Scalar Equation of a Plane

Imagine a plane containing point \( P(x_p, y_p, z_p) \), which is known, and a general point \( Q(x_q, y_q, z_q) \).

The vector \( \vec{PQ} = (x_q-x_p, y_q-y_p, z_q-z_p) \) represents a vector in the plane.

Let \( \vec{n} = (A, B, C) \) be a known normal to the plane.

According to the dot product, \( \vec{n} \cdot \vec{PQ} = 0 \).

\[
(A, B, C) \cdot (x_q-x_p, y_q-y_p, z_q-z_p) = 0
\]

\[
Ax_q - Ax_p + By_q - By_p + Cz_q - Cz_p = 0
\]

Since all values in \( -(Ax_p + By_p + Cz_p) \) are known, replace it with a constant \( D \).

The scalar equation of a plane, with normal vector \( \vec{n} = (A, B, C) \), is

\[
Ax + By + Cz + D = 0
\]

**Example**

Determine the scalar equation of the plane with normal \( \vec{n} = (3, -2, 5) \) that contains the point \( P(1, 0, -1) \).

\[
3(1) - 2(0) + 5(-1) + D = 0
\]

\[
-2 + D = 0
\]

\[
D = 2
\]

The scalar equation is \( 3x - 2y + 5z + 2 = 0 \).

Scalar Equation of a Plane

Use \( \vec{n} \) and a point in the plane to find the scalar equation.

\[
10(1) - 2(0) + 7(3) + D = 0
\]

\[
31 + D = 0
\]

\[
D = -31
\]

The scalar equation is \( 10x - 2y + 7z - 31 = 0 \).
Scalar Equation of a Plane

Example

Determine the scalar equation of the plane with vector equation \( \mathbf{r} = (3,1, -2) + s(1,0,2) + t(-2,1,0) \).

Two direction vectors are given in the vector equation, so find the cross product of them to determine the normal to the plane.

\[ \mathbf{n} = (1,0,2) \times (-2,1,0) = (-2,-1,4) \].

Use the normal and a point on the line to determine the scalar equation.

\[
-2(3) - 4(1) + 1(-2) + D = 0 \\
D = 12
\]

The scalar equation is \( 2x + 4y - z = 12 \).

Scalar Equation of a Plane

Two direction vectors for the line are \( \mathbf{PQ} = (0,1,2) \) and \( \mathbf{QR} = (1,-1,1) \).

A possible vector equation for the line is \( \mathbf{r} = (1,0,-2) + s(0,1,2) + t(1,-1,1) \).

The corresponding parametric equations are \( x = 1 + t \), \( y = s - t \) and \( z = -2 + 2s + t \).

Scalar Equation of a Plane

If two planes are parallel (or coincident), their normals will also be parallel.

This implies that \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \).

This means that \( \mathbf{n}_1 = k\mathbf{n}_2 \) for some real value \( k \).

If two planes are perpendicular, their normals will also be perpendicular.

This implies that \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \).

Scalar Equation of a Plane

Example

Determine if the planes \( \pi_1 : x - 3y + 5z + 4 = 0 \) and \( \pi_2 : 4x - 12y + 20z + 18 = 0 \) are coincident, parallel, perpendicular or neither.

From the given equations, \( \mathbf{n}_1 = (1,-3,5) \) and \( \mathbf{n}_2 = (4,-12,20) \). Since \( \mathbf{n}_2 = 4\mathbf{n}_1 \), the planes are either coincident or parallel.

Since the equation for \( \pi_2 \) is not a scalar multiple of the equation for \( \pi_1 \) (the constant is off), the planes are not coincident. Therefore, planes \( \pi_1 \) and \( \pi_2 \) are parallel.
Scalar Equation of a Plane

Two non-parallel planes must intersect (more on this later).

The angle formed between the two planes will be the same as the angle formed between their normals.

We can use the dot product to calculate the measure of this angle.

**Angle Between Two Intersecting Planes**

The angle, \( \theta \), between two intersecting planes \( \pi_1 \) and \( \pi_2 \) is given by

\[
\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}.
\]

Example

Determine the angle formed between the intersecting planes \( \pi_1 : x - y - 2z + 3 = 0 \) and \( \pi_2 : 2x + y - z + 2 = 0 \).

From the given equations, \( \vec{n}_1 = (1, -1, -2) \) and \( \vec{n}_2 = (2, 1, -1) \).

Calculate the magnitudes of \( \vec{n}_1 \) and \( \vec{n}_2 \).

\[
|\vec{n}_1| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}
\]

\[
|\vec{n}_2| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}
\]

Use the dot product to calculate the angle between the normals and, thus, between the planes.

\[
\cos \theta = \frac{(1, -1, -2) \cdot (2, 1, -1)}{\sqrt{6} \sqrt{6}} = \frac{2 - 1 + 2}{6} = \frac{1}{2}
\]

Therefore, the angle between the planes is \( 60^\circ \) (or \( 120^\circ \)).

Questions?