MCV4U: Calculus & Vectors

Related Rates

Part 1

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Related Rates

Related rates are usually one of the trickier concepts in high school calculus.

Typically, we will want to determine (or evaluate) a rate of change for some quantity, which is based on one or more other quantities.

Since related rates often involve one quantity, y, that depends on another, u, which is based on a third quanity, x, the chain rule is used.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

A good strategy is to list all given quantities, as well as those that we need to find, and try to find links between them.

Related Rates

Example

A spherical balloon is being filled with helium at a rate of 10 cm^3/s. At what rate is the radius increasing when the radius is 5 cm?

We want to know the change in the balloon's radius with respect to time, or $\frac{dr}{dt}$.

We are given the change in volume with respect to time, $\frac{dV}{dt} = 10$.

The volume of the balloon is given by $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$.

Since the radius of the balloon is 5 cm, $\frac{dV}{dr}\Big|_{r=5} = 4\pi(5)^2 = 100\pi$.

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Putting all of this together,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$
$$10 = 100\pi \cdot \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{10\pi}$$

Therefore, the radius is increasing at a rate of $\frac{1}{10\pi}$ cm/s.

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Example

An ice cube melts without changing shape at a uniform rate of 4 cm^3/min. What is the rate of change of its surface area when the volume of the cube is 125 cm^3?

We want to know the change in the cube's surface area with respect to time, or $\frac{dA}{dt}$.

We are given the change in volume with respect to time, $\frac{dV}{dt} = -4$. Note that the value is negative, since the volume is decreasing.

The volume of the cube is given by $V = s^3$, so $\frac{dV}{ds} = 3s^2$.

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Using the chain rule,

$$\frac{dV}{dt} = \frac{dV}{ds} \cdot \frac{ds}{dt}$$
$$-4 = 3s^2 \cdot \frac{ds}{dt}$$
$$\frac{ds}{dt} = -\frac{4}{3s^2}$$

The surface area of the cube is given by $A = 6s^2$, so $\frac{dA}{ds} = 12s$.

Using the chain rule again,

$$\frac{dA}{dt} = \frac{dA}{ds} \cdot \frac{ds}{dt}$$
$$= 12s\left(-\frac{4}{3s^2}\right)$$
$$= -\frac{16}{s}$$

When the volume is 125 cm^3, then $s = 5$, so $\frac{dA}{dt}\Big|_{s=5} = -\frac{16}{5} = -3.2$ cm^2/s.

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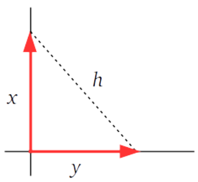
Example

Two students leave school at the same time. Gabriella walks north at 1.8 m/s, while Alexander walks east at 1.2 m/s. How fast is the distance between them changing after 5 minutes?

The two paths form a right triangle as shown, where x is the distance Gabriella walks and y the distance Alexander walks.



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After 5 minutes, Gabriella has walked $300 \cdot 1.8 = 540$ m, while Alexander has walked $300 \cdot 1.2 = 360$ m.

We are given both $\frac{dx}{dt} = 1.8$ and $\frac{dy}{dt} = 1.2$, and we want to know $\frac{dh}{dt}$.

Using implicit differentiation,

$$\begin{aligned} x^2 + y^2 &= h^2 \\ 2x\tfrac{dx}{dt} + 2y\tfrac{dy}{dt} &= 2h\tfrac{dh}{dt} \\ 2 \cdot 540 \cdot 1.8 + 2 \cdot 360 \cdot 1.2 &= 2h\tfrac{dh}{dt} \\ 2808 &= 2h\tfrac{dh}{dt} \\ \tfrac{dh}{dt} &= \frac{2808}{2h} \end{aligned}$$

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Using the Pythagorean Theorem, we can calculate h at 5 minutes.

$$\begin{aligned} h &= \sqrt{540^2 + 360^2} \\ &= 180\sqrt{13} \end{aligned}$$

Using this value,

$$\begin{aligned} \tfrac{dh}{dt} &= \frac{2808}{2 \cdot 180\sqrt{13}} \\ &= \tfrac{3\sqrt{13}}{5} \end{aligned}$$

Therefore, the distance between Gabriella and Alexander is increasing by approximately 2.16 m/s at 5 minutes.

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Alternatively, we can solve the problem by expanding instead.

Since Gabriella walks at a rate of 1.8 m/s, and Alexander walks at 1.2 m/s, expressions for their distances travelled are $1.8t$ and $1.2t$ respectively.

By the Pythagorean Theorem,

$$\begin{aligned} h^2 &= (1.8t)^2 + (1.2t)^2 \\ &= 3.24t^2 + 1.44t^2 \\ &= 4.68t^2 \end{aligned}$$

Using implicit differentiation,

$$\begin{aligned} 2h\tfrac{dh}{dt} &= 9.36t \\ \tfrac{dh}{dt} &= \frac{9.36t}{2h} \end{aligned}$$

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Using $h = 180\sqrt{13}$ and $t = 300$ as before, evaluate $\frac{dh}{dt}$.

$$\begin{aligned} \tfrac{dh}{dt} &= \frac{9.36 \cdot 300}{2 \cdot 180\sqrt{13}} \\ &= \tfrac{3\sqrt{13}}{5} \\ &\approx 2.16 \text{ m/s} \end{aligned}$$

Either method is acceptable. In some cases, one method is easier than another.

Questions?