MCV4U: Calculus & Vectors

# Quotient Rule of Derivatives

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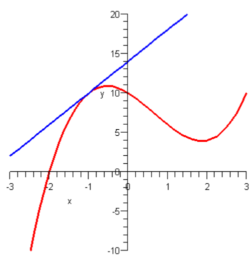







## Chain Rule

**Recap**

Determine the derivative of $f(x) = 12\sqrt[3]{2x^5 - 7x}$.

The inner function is $h(x) = 2x^5 - 7x$, so $h'(x) = 10x^4 - 7$.

The outer function is $g(h) = 12h^{\frac{1}{3}}$, so $g'(x) = 4h^{-\frac{2}{3}}$.

$$f'(x) = 4(2x^5 - 7x)^{-\frac{2}{3}}(10x^4 - 7x)$$

$$\text{or } \frac{40x^4 - 28}{(x(2x^4 - 7))^{\frac{2}{3}}}$$





## Quotient Rule

Sometimes a function can be written as the quotient of two other functions.

To determine the derivative of a function with the form $f(x) = \frac{p(x)}{q(x)}$, use the Quotient Rule.

**Quotient Rule**

If $f(x) = \frac{p(x)}{q(x)}$, then $f'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{[q(x)]^2}$, provided that $q(x) \neq 0$.

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, $v \neq 0$.

Note the similarity to the product rule in the numerator.





## Quotient Rule

Rewrite $f(x) = \frac{p(x)}{q(x)}$ as $f(x) = p(x)[q(x)]^{-1}$ and use the chain rule to differentiate.

$$\begin{aligned} f'(x) &= p'(x)[q(x)]^{-1} + p(x)(-1)[q(x)]^{-2}q'(x) \\ &= \left(p'(x)[q(x)]^{-1} - p(x)[q(x)]^{-2}q'(x)\right) \times \frac{[q(x)]^2}{[q(x)]^2} \\ &= \frac{p'(x)q(x) - p(x)q'(x)}{[q(x)]^2} \end{aligned}$$





## Quotient Rule

**Example**

Determine the derivative of $f(x) = \frac{3x^2 - 1}{2x + 3}$.

Let $p(x) = 3x^2 - 1$ and $q(x) = 2x + 3$.

Then $p'(x) = 6x$ and $q'(x) = 2$.

$$\begin{aligned} f'(x) &= \frac{(6x)(2x+3) - (3x^2 - 1)(2)}{(2x+3)^2} \\ &= \frac{12x^2 + 18x - 6x^2 + 2}{(2x+3)^2} \\ &= \frac{6x^2 + 18x + 2}{(2x+3)^2} \end{aligned}$$





## Quotient Rule

**Example**

Determine the derivative of $y = \frac{4x^2 - 6}{\sqrt{x}}$.

Let $u = 4x^2 - 6$ and $v = \sqrt{x} = x^{\frac{1}{2}}$.

Then $\frac{du}{dx} = 8x$ and $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$.

$$\frac{dy}{dx} = \frac{(8x)(\sqrt{x}) - (4x^2 - 6)\left(\frac{1}{2\sqrt{x}}\right)}{[\sqrt{x}]^2}$$

$$\text{or } \frac{6x^2 + 3}{\sqrt{x^3}}$$

## Quotient Rule

**Example**

The concentration, $C$ (g/L), of a chemical released into a stream after $t$ days is given by $C(t) = \frac{6t}{2t^2+9}$. At what rate is the concentration changing after 4 days?

Let $p(t) = 6t$ and $q(t) = 2t^2 + 9$.

Then $p'(t) = 6$ and $q'(t) = 4t$.

$$\begin{aligned} f'(x) &= \frac{6(2t^2+9) - 6t(4t)}{(2t^2+9)^2} \\ &= \frac{6(2(4)^2+9) - 6(4)(4(4))}{(2(4)^2+9)^2} \\ &= -\frac{138}{1681} \end{aligned}$$





## An Alternative to the Quotient Rule

Since we derived the general formula for the quotient rule using the chain rule, an alternative to the quotient rule is to use both the product rule and the chain rule instead.

This may be useful, because it is not necessary to memorize a separate formula for quotients.

Derivatives derived using each technique may appear different, but should be the same after simplification.





## An Alternative to the Quotient Rule

**Example**

Use the product and chain rules to show that the derivative of $f(x) = \frac{3x^2-1}{2x+3}$ is $f'(x) = \frac{6x^2+18x+2}{(2x+3)^2}$.

Rewrite the function as $f(x) = (3x^2-1)(2x+3)^{-1}$.

$$\begin{aligned} f'(x) &= (6x)(2x+3)^{-1} + (3x^2-1)(-1)(2x+3)^{-2}(2) \\ &= (6x)(2x+3)^{-1} - 2(3x^2-1)(2x+3)^{-2} \\ &= \frac{(6x)(2x+3) - 6x^2 + 2}{(2x+3)^2} \\ &= \frac{6x^2+18x+2}{(2x+3)^2} \end{aligned}$$





## An Alternative to the Quotient Rule

**Example**

At what point(s), if any, is the tangent to $y = \frac{3x-4}{x^2-5x}$ horizontal?

Rewrite the function as $y = (3x-4)(x^2-5x)^{-1}$.

The slope of the tangent is given by $\frac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= 3(x^2-5x)^{-1} + (3x-4)(-1)(x^2-5x)^{-2}(2x-5) \\ &= \frac{3(x^2-5x) - (3x-4)(2x-5)}{(x^2-5x)^2} \\ &= \frac{-3x^2+8x-20}{(x^2-5x)^2} \end{aligned}$$





## An Alternative to the Quotient Rule

Since the slope of a horizontal line is 0, set $\frac{dy}{dx} = 0$ and solve for $x$.

$$\begin{aligned} 0 &= \frac{-3x^2+8x-20}{(x^2-5x)^2} \\ &= -3x^2+8x-20 \end{aligned}$$

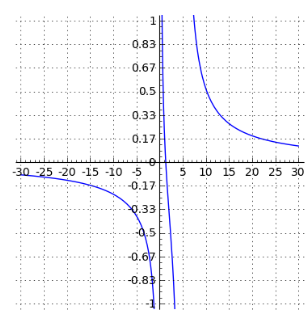
Since $8^2 - 4(-3)(-20) < 0$, there are no real solutions to this equation.

Therefore, there are no points on $y = \frac{3x-4}{x^2-5x}$ where the tangent is horizontal.





## An Alternative to the Quotient Rule



## Questions?