MCV4U: Calculus & Vectors

Limits and Their Properties

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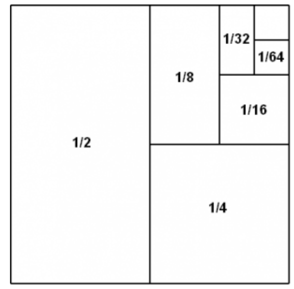
Limits

What is the value of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$?

A visual depiction is below.



Limits

If we use the first four terms of the sequence, then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$.

If we increase the number of terms, we obtain the following:

Terms	5	6	7	...	20
Sum	$\frac{31}{32}$	$\frac{63}{64}$	$\frac{127}{128}$	...	$\frac{1048575}{1048576}$

As the number of terms increases, the sum approaches 1. We call this concept a *limit*.

Limits

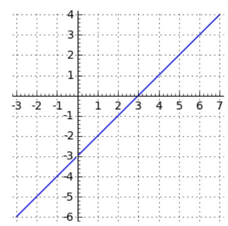
A limit is some value that a function (or sequence) approaches, as the input (or index) approaches some value.

Limits

Example

Determine the value of $\lim_{x \to 4} (x - 3)$.

This is a linear function, $f(x) = x - 3$, whose graph is below.



Limits

The expression $\lim_{x \to 4} (x - 3)$ means "what value does the linear function approach as x gets closer to 4?"

By observation, as $x \to 4$, $f(x) \to 1$.

Therefore, we state that $\lim_{x \to 4} (x - 3) = 1$.

In this example, it is also true that $f(4) = 1$, but this does not always need to be true.

Limits

Example

Determine the value of $\lim_{x \to \infty} \frac{1}{x}$.

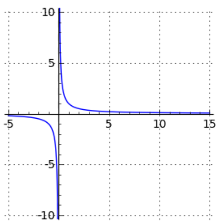
This time, we are not approaching a specific *value*, but ∞ itself.

Recall that the end behaviour of the function $\frac{1}{x}$ is defined by values of x that approach ∞.

Thus, the question can be restated as "does the end behaviour of $f(x) = \frac{1}{x}$ cause it to approach a specific value?"

Again, a graph of the function (or a knowledge of its basic properties) is useful.

Limits



The graph of $f(x) = \frac{1}{x}$ has a horizontal asymptote at $f(x) = 0$, and as $x \to \infty$, $f(x) \to 0$.

While the function never actually takes on a value of 0, it gets *infinitesimally close* to 0 and we say that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Limits

Example

Determine the value of $\lim_{x \to 0} \frac{x-3}{x^2}$.

While it is possible to graph this rational function, an alternative method is to use a table of values that become closer and closer to 0.

First, check values that are less than 0.

x	-0.1	-0.01	-0.001	...
$f(x)$	-310	-30100	-3×10^6	...

$f(x)$ decreases rapidly, the closer it gets to 0. It appears that as $x \to 0$, $f(x) \to -\infty$.

Limits

Next, check values that are greater than 0.

x	...	0.001	0.01	0.1
$f(x)$	...	-3×10^6	-30100	-310

Again, $f(x)$ decreases rapidly and as $x \to 0$, $f(x) \to -\infty$.

Since all values suggest that $f(x)$ continues to decrease the closer it gets to 0, we say that $\lim_{x \to 0} \frac{x-3}{x^2} = -\infty$.

Remember that ∞ is not a value. A function can approach ∞, but will never reach it!

One-Sided Limits

In the last example, we tested values close to a specific value.

A limit that approaches a certain value "from the left" or "from the right" is called a *one-sided limit*.

Left- and Right-Handed Limits

For a function $f(x)$, we denote as follows:

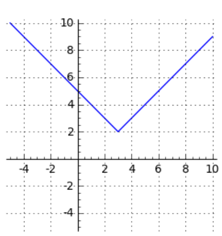
- the limit as x approaches a from the left, $\lim_{x \to a^-} f(x)$, is called the *left-handed limit*
- the limit as x approaches a from the right, $\lim_{x \to a^+} f(x)$, is called the *right-handed limit*

The values of the left- and right-handed limits may be different, depending on the function, or they may not exist at all.

One-Sided Limits

Example

For the function below, state the values of $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$, if they exist.



One-Sided Limits

Moving from the left, $\lim_{x \to 3^-} f(x) = 2$.

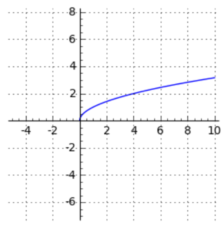
Moving from the right, $\lim_{x \to 3^+} f(x) = 2$.

In this case, the left- and right-handed limits are equal. This is not always true.

One-Sided Limits

Example

For the function below, state the values of $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$, if they exist.



One-Sided Limits

Moving from the right, $\lim_{x \to 0^+} f(x) = 0$.

It is not possible to approach 0 from the left, however, since $f(x)$ is not defined for any $x < 0$.

Therefore, $\lim_{x \to 0^-} f(x)$ does not exist.

Limit of a Function

We can define the limit of a function using left- and right-handed limits.

Limit of a Function

Given a function $f(x)$, the limit as $x \to a$ exists if the left- and right-handed limits exist and are equal. Mathematically, $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

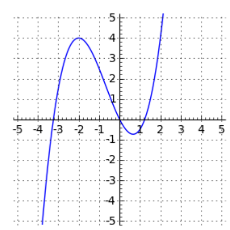
Using this definition, the limit of $f(x)$ as $x \to 0$ in the previous example does not exist, since $\lim_{x \to 0^-} f(x)$ does not exist.

A more formal definition of limits involving small quantities δ and ϵ is typically covered in first-year university courses, but this definition will suit us for now.

Limit of a Function

Example

Determine $\lim_{x \to 2} f(x)$ for $f(x)$ below, if it exists.

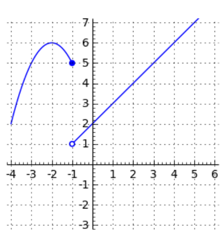


Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 4$, then $\lim_{x \to 2} f(x) = 4$.

Limit of a Function

Example

Determine $\lim_{x \to -1} f(x)$ for $f(x)$ below, if it exists.

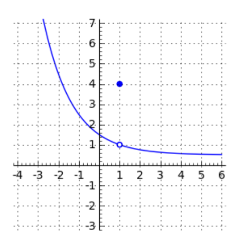


Since $\lim_{x \to -1^-} f(x) = 5$ and $\lim_{x \to -1^+} f(x) = 1$, then $\lim_{x \to -1} f(x)$ does not exist.

Limit of a Function

Example

Determine $\lim_{x \to 1} f(x)$ for $f(x)$ below, if it exists.



Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1$, then $\lim_{x \to 1} f(x) = 1$.

Properties of Limits

Note that $f(x)$ is *discontinuous* at $x = 1$, and that $f(1) = 4$. We will talk about this in more detail shortly.

For now, just remember that the limit of a function as x approaches some value does not always need to have the same value as the function at that value.

While drawing a graph or creating a table of values is useful for simple functions, there are algebraic properties of limits that make it possible to evaluate them without these tools.

Most of these are fairly easy to prove, if you are so inclined, but proofs are not provided here.

Properties of Limits

Limit Properties

1. $\lim_{x \to a} c = c$ if c is a constant.
2. $\lim_{x \to a} x = a$.
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ for some constant c.
4. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$.
5. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$.
6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ where $\lim_{x \to a} g(x) \neq 0$.
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ where n is rational.

Properties of Limits

Example

Use limit properties to evaluate $\lim_{x \to 2} (3x - 5)$.

$$\begin{aligned} \lim_{x \to 2} (3x - 5) &= \lim_{x \to 2} 3x - \lim_{x \to 2} 5 && (4) \\ &= 3 \lim_{x \to 2} x - \lim_{x \to 2} 5 && (3) \\ &= 3(2) - 5 && (1 \text{ and } 2) \\ &= 1 \end{aligned}$$

Properties of Limits

Example

Use limit properties to evaluate $\lim_{x \to 5} \sqrt{3x + 1}$.

$$\begin{aligned} \lim_{x \to 5} \sqrt{3x+1} &= \lim_{x \to 5} (3x+1)^{\frac{1}{2}} \\ &= \left[\lim_{x \to 5} (3x+1)\right]^{\frac{1}{2}} && (7) \\ &= \left[3 \lim_{x \to 5} x + \lim_{x \to 5} 1\right]^{\frac{1}{2}} && (3 \text{ and } 4) \\ &= [3(5) + 1]^{\frac{1}{2}} && (1 \text{ and } 2) \\ &= 4 \end{aligned}$$

Questions?