MCV4U: Calculus & Vectors

Product Rule of Derivatives

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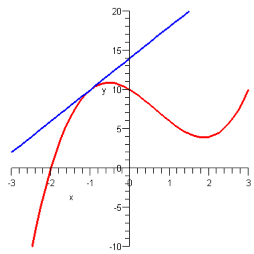

Product Rule

Recap

Determine the derivative of $f(x) = (x^2 - 3)(x^5 + 2x)$.

Expand the polynomial, then use the basic derivative rules discussed earlier.

$$f(x) = x^7 - 3x^5 + 2x^3 - 6x$$
$$f'(x) = 7x^6 - 15x^4 + 6x^2 - 6$$

Product Rule

Since $f(x)$ is the product of two binomials, we can write $f(x) = g(x)h(x)$.

Can we find $f'(x)$ using the product of $g'(x)$ and $h'(x)$?

Let $g(x) = x^2 - 3$ and $h(x) = x^5 + 2x$.

Then $g'(x) = 2x$ and $h'(x) = 5x^4 + 2$.

$g'(x)h'(x) = 10x^5 + 4x$, which is not the same as $f'(x)$.

To determine the derivative of a function that is the product of two other functions, we must use the Product Rule.

Product Rule

If $f(x) = p(x)q(x)$, then $f'(x) = p'(x)q(x) + p(x)q'(x)$.

If $y = uv$, then $\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$.

Product Rule

$$f'(x) = \lim_{h\to 0} \frac{p(x+h)q(x+h) - p(x)q(x)}{h}$$

Let $k = q(x+h)$ so the proof fits nicely on this slide.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{p(x+h)k - p(x)q(x)}{h} \\ &= \lim_{h\to 0} \frac{p(x+h)k - p(x)k + p(x)k - p(x)q(x)}{h} \\ &= \lim_{h\to 0} \left(\left[\frac{p(x+h)-p(x)}{h}\right] k + p(x)\left[\frac{k-q(x)}{h}\right]\right) \\ &= \lim_{h\to 0} \left(\left[\frac{p(x+h)-p(x)}{h}\right] k \right) + \lim_{h\to 0}\left(p(x)\left[\frac{k-q(x)}{h}\right]\right) \end{aligned}$$

Product Rule

With that done, substitute $k = q(x+h)$ back.

$$\begin{aligned} &= \lim_{h\to 0} \left(\left[\frac{p(x+h)-p(x)}{h}\right] q(x+h) \right) \\ &\quad + \lim_{h\to 0}\left(p(x)\left[\frac{q(x+h)-q(x)}{h}\right]\right) \\ &= \lim_{h\to 0} \left[\frac{p(x+h)-p(x)}{h}\right] \lim_{h\to 0} q(x+h) \\ &\quad + \lim_{h\to 0} p(x) \lim_{h\to 0} \left[\frac{q(x+h)-q(x)}{h}\right] \\ &= p'(x)q(x) + p(x)q'(x) \end{aligned}$$

Product Rule

Example

Use the product rule to confirm that the derivative of $f(x) = (x^2 - 3)(x^5 + 2x)$ is $f'(x) = 7x^6 - 15x^4 + 6x^2 - 6$.

Let $p(x) = x^2 - 3$ and $q(x) = x^5 + 2x$.

Then $p'(x) = 2x$ and $q'(x) = 5x^4 + 2$.

$$f'(x) = (2x)(x^5 + 2x) + (x^2 - 3)(5x^4 + 2)$$

While this result looks different, it can be expanded and simplified.

$$\begin{aligned} f'(x) &= (2x^6 + 4x^2) + (5x^6 - 15x^4 + 2x^2 - 6) \\ &= 7x^6 - 15x^4 + 6x^2 - 6 \end{aligned}$$

Usually, expanding takes longer than determining the derivative. Don't do so unless it is necessary.

Product Rule

Example

Determine the derivative of $y = (4x^3 + 5x)(2x^5 - 3x^2)$.

Let $u = 4x^3 + 5x$ and $v = 2x^5 - 3x^2$.

Then $\frac{du}{dx} = 12x^2 + 5$ and $\frac{dv}{dx} = 10x^4 - 6x$.

$$\frac{dy}{dx} = (12x^2 + 5)(2x^5 - 3x^2) + (4x^3 + 5x)(10x^4 - 6x)$$
$$\text{or } 64x^7 + 60x^5 - 60x^4 - 45x^2$$

Of course, the derivative *could* have been found by expanding first. There are some functions, however, for which the product rule is the only option.

Product Rule

Example

Determine the slope of the tangent of $f(x) = (2x^2 - 3x + 5)(4x^3 - x + 2)$ when $x = -1$.

This is far too tedious to expand, so the product rule is much quicker in this case.

$$f'(x) = (4x - 3)(4x^3 - x + 2) + (2x^2 - 3x + 5)(12x^2 - 1)$$

Substitute $x = -1$ into $f'(x)$.

$$\begin{aligned} f'(-1) =& (4(-1) - 3)(4(-1)^3 - (-1) + 2) \\ &+ (2(-1)^2 - 3(-1) + 5)(12(-1)^2 - 1) \\ =& 117 \end{aligned}$$

The slope of the tangent is 117.

Extending the Product Rule

We can extend the product rule to cover cases where $f(x) = p(x)q(x)r(x)$.

Group $p(x)$ and $q(x)$ such that $f(x) = [p(x)q(x)]r(x)$.

Applying the product rule, we obtain

$$f'(x) = [p(x)q(x)]'r(x) + [p(x)q(x)]r'(x)$$

Apply the product rule again to obtain

$$\begin{aligned} f'(x) &= [p'(x)q(x) + p(x)q'(x)]r(x) + [p(x)q(x)]r'(x) \\ &= p'(x)q(x)r(x) + p(x)q'(x)r(x) + p(x)q(x)r'(x) \end{aligned}$$

This pattern can be extended to determine the derivative of a function that is the product of any number of factors.

Extending the Product Rule

Example

Determine the derivative of $f(x) = (3x - 2)(4x + 1)(6x^2 - x)$.

Let $p(x) = 3x - 2$, $q(x) = 4x + 1$, $r(x) = 6x^2 - x$.

Then $p'(x) = 3$, $q'(x) = 4$ and $r'(x) = 12x - 1$.

$$\begin{aligned} f'(x) &= 3(4x + 1)(6x^2 - x) + (3x - 2)(4)(6x^2 - x) \\ &\quad + (3x - 2)(4x + 1)(12x - 1) \\ &\quad \text{or } 288x^3 - 126x^2 - 14x + 2 \end{aligned}$$

Extending the Product Rule

Example

Determine the derivative of $f(x) = (2x - 5)^3$.

Let $p(x) = q(x) = r(x) = 2x - 5$.

Then $p'(x) = q'(x) = r'(x) = 2$.

$$\begin{aligned} f'(x) &= 2(2x - 5)^2 + 2(2x - 5)^2 + 2(2x - 5)^2 \\ &= 6(2x - 5)^2 \\ &\text{or } 24x^2 - 120x + 150 \end{aligned}$$

This last example is a generalization of the Chain Rule, which we will discuss later.

Applying the Product Rule

Example

A company typically sells 200 t-shirts each month for \$15 a piece. For each increase of \$1, the company will lose 10 sales per month.

- Determine an equation to model the revenue, R, as a function of n, the number of price increases.
- Determine the rate of change in revenue when the cost of a t-shirt is \$20.
- Solve for $R'(n) = 0$ and explain what it represents.

Applying the Product Rule

Revenue is units sold × unit cost.

$$R(n) = (200 - 10n)(15 + n)$$

The cost of the t-shirt is \$20 after 5 increases of \$1.

$$\begin{aligned} R'(n) &= (-10)(15+n) + 1(200-10n) \\ &= 50 - 20n \\ R'(5) &= 50 - 20(5) \\ &= -50 \end{aligned}$$

Therefore, the company is *losing* revenue at a rate of \$50 per price increase.

Applying the Product Rule

$R'(n) = 0$ when $n = \frac{5}{2}$. This corresponds to a price increase of \$2.50 per shirt.

Since the revenue function is a quadratic, opening downward, $R'(n) = 0$ at the vertex. This means that the maximum revenue is obtained when the company sells the shirts for \$17.50.

The same answer could have been obtained by completing the square or by using partial factoring, but often setting the derivative equal to zero is faster.

Questions?