MCV4U: Calculus & Vectors

Optimization
Part 2

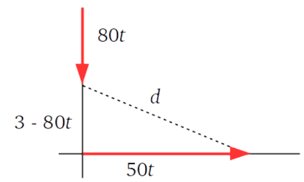
J. Garvin

Optimization

Example

A car is travelling east along an east-west road at 50 km/h. As it passes though an intersection, a second car 3 km north of the intersection is travelling south at 80 km/h. When are the cars closest together, and how far apart are they?

The diagram below shows the distance, d, between the cars.



Optimization

We might begin by finding an equation for the distance between the cars, using the Pythagorean Theorem.

$$D = \sqrt{(50t)^2 + (3 - 80t)^2}$$

This will work but will involve the chain rule, making our calculations more difficult.

While we want to minimize the distance between the two cars, we can instead minimize the *square* of the distance instead, eliminating the radical completely.

This should make sense, since the the minimum value in a set of numbers will have a square that is also a minimum among all squares of those numbers.

Optimization

Therefore, an equation for the square of the distance between the cars after t hours is,

$$\begin{aligned} D &= (50t)^2 + (3 - 80t)^2 \\ &= 8900t^2 - 480t + 9 \end{aligned}$$

Find the derivative to identify critical points.

$$\frac{dD}{dt} = 17800t - 480$$

A critical point is at $t = \frac{480}{17800} = \frac{12}{445}$ h.

Optimization

Since D is a quadratic relation with a positive leading coefficient, this point is a local minimum at the vertex. This can, of course, be verified using either the first or second derivative.

At $\frac{12}{445}$ h, $D = \frac{225}{89}$. So $d = \sqrt{\frac{225}{89}} = \frac{15}{\sqrt{89}}$.

Therefore, the cars are closest at $\frac{12}{445} \approx 0.027$ h, or 1.62 min. At this time, they are $\frac{15}{\sqrt{89}} \approx 1.59$ km apart.

Optimization

Example

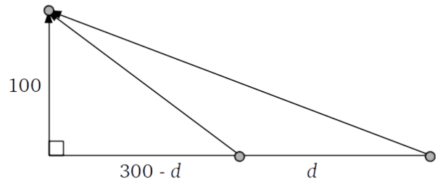
A individual can swim with a speed of 1.2 m/s, and run on land with a speed of 2.5 m/s. A buoy floats 100 m from shore, at a point 300 m downstream. Where should the individual enter the water to reach the buoy fastest?

There are three possibilities. The individual may:

- enter the water immediately and swim the entire distance straight to the buoy;
- run downstream to a point opposite the buoy, then swim directly to it; or
- run some distance d downstream, then swim directly toward the buoy.

Optimization

The three paths are shown below.



Unfortunately, since we are not minimizing the distance, we cannot use the previous technique of using the square of the distance and, as such, the calculations will be messier.

Optimization

Using the Pythagorean Theorem, the distance travelled entirely through the water is $\sqrt{300^2+100^2} = 100\sqrt{10}$ m. Swimming this distance at 1.2 m/s would take a total of $\frac{100\sqrt{10}}{1.2} \approx 263.5$ s.

Running to a point opposite the buoy then swimming through the water covers a distance of 300 m at 2.5 m/s and 100 m at 1.2 m/s. The total time is $\frac{300}{2.5} + \frac{100}{1.2} \approx 203.3$ s, which is faster than swimming directly to the buoy.

The last of the three options requires a bit more work.

Optimization

The last option sees the individual run d m on land at a speed of 2.5 m/s. This portion takes a total of $\frac{d}{2.5}$ s.

Once the individual has run d m, s/he needs to swim directly to the buoy. Using the Pythagorean Theorem, this distance is $\sqrt{(300-d)^2+100^2}$ m.

At 1.2 m/s, this portion of the trip takes a total of $\frac{\sqrt{(300-d)^2+100^2}}{1.2}$ s.

Therefore, an equation for the total time is

$$\begin{aligned} T &= \frac{d}{2.5} + \frac{\sqrt{(300-d)^2+100^2}}{1.2} \\ &= \tfrac{2}{5}d + \tfrac{5}{6}\sqrt{d^2-600d+100000} \end{aligned}$$

Optimization

Determine the derivative to look for critical points.

$$\begin{aligned} \frac{dT}{dd} &= \tfrac{2}{5} + \tfrac{5}{6} \cdot \tfrac{1}{2}(d^2-600d+100000)^{-1/2}(2d-600) \\ &= \tfrac{2}{5} + \frac{5(2d-600)}{12\sqrt{d^2-600d+100000}} \\ &= \frac{24\sqrt{d^2-600d+100000}+25(2d-600)}{60\sqrt{d^2-600d+100000}} \end{aligned}$$

Set the derivative equal to zero and solve for d. Remember that the denominator will cancel out.

$$24\sqrt{d^2-600d+100000}+25(2d-600)=0$$

Optimization

$$\begin{aligned} 24\sqrt{d^2-600d+100000} &= -25(2d-600) \\ 576(d^2-600d+100000) &= 625(2d-600)^2 \\ 576d^2-345600d+57600000 &= 2500d^2-1500000d+225000000 \\ 1924d^2-1154400d+167400000 &= 0 \end{aligned}$$

Use the quadratic formula.

$$\begin{aligned} d &= \frac{1154400 \pm \sqrt{(-1154400^2)-4(1924)(167400000)}}{2(1924)} \\ &\approx 245.28 \text{ or } 354.72 \end{aligned}$$

Optimization

Since 354.72 is outside of the domain ($d > 300$), 245.28 is the only solution representing the distance ran along the shore.

If $d \approx 245.28$, then the distance swam is $\sqrt{(300-245.28)^2+100^2} \approx 114.0$ m.

Running 245.28 m along shore will take a total of $\frac{245.28}{2.5} \approx 98.1$ s, while swimming 114.0 m in the water will take $\frac{114.0}{1.2} \approx 95$ s, for a total of 193.1 s.

This is faster than either of the previous two options, by just over 10 seconds.

Questions?