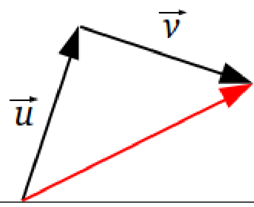
MCV4U: Calculus & Vectors

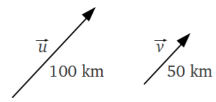
Multiplying Vectors By Scalars

J. Garvin



Multiplying a Vector By a Scalar

Compare the two vectors, $\vec{u}$ and $\vec{v}$.



$\vec{u}$ and $\vec{v}$ have the same direction, but different magnitudes.

In this case, $\vec{u}$ is twice as long as $\vec{v}$.

In terms of their magnitudes, $|\vec{u}| = 2|\vec{v}|$ or $|\vec{v}| = \frac{1}{2}|\vec{u}|$.

Using the vectors themselves, $\vec{u} = 2\vec{v}$, or $\vec{v} = \frac{1}{2}\vec{u}$.

Multiplying a Vector By a Scalar

It is possible to enlarge or reduce the magnitude of a vector by some constant factor.

Scalar Multiplication

Given vector $\vec{v}$ and scalar k, $k\vec{v}$ is a vector that is $|k|$ times as long.

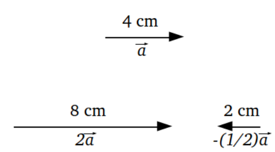
Note that the direction may change, depending on the sign of the scalar k.

- If $k > 0$, $k\vec{v}$ has the same direction as $\vec{v}$.
- If $k < 0$, $k\vec{v}$ has the opposite direction as $\vec{v}$.
- If $k = 0$, $k\vec{v}$ is the zero vector.

Multiplying a Vector By a Scalar

Example

Given vector $\vec{a}$, draw $2\vec{a}$ and $-\frac{1}{2}\vec{a}$.



Collinear Vectors

Two vectors that form a straight line are *collinear*.

Collinear Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are collinear if, and only if, there is some non-zero scalar k such that $\vec{u} = k\vec{v}$.

Reason: If two vectors $\vec{u}$ and $\vec{v}$ are collinear, they are parallel. This means that either $\vec{u}$ and $\vec{v}$ have the same direction, or they have opposite directions.

If they have the same direction, but different magnitudes, then $k > 0$.

If they have opposite directions, and different magnitudes, then $k < 0$.

If the magnitudes are the same, then $k = 1$ or $k = -1$, depending on the direction.

Collinear Vectors

Example

Vector $\vec{a}$ has a magnitude of 5, with a bearing of 315°. Describe a vector $\vec{b}$ that is collinear to $\vec{a}$ if it has the opposite direction and the same magnitude.

Since only the direction is reversed, $k = -1$.

This produces the vector $\vec{b}$ such that $|\vec{b}| = 5$ with a bearing of 135° (since $135° + 180° = 315°$).

Therefore, $\vec{b} = -\vec{a}$.

Collinear Vectors

Example

Vector $\vec{a}$ has a magnitude of 5, with a bearing of 315°. Describe a vector $\vec{b}$ that is collinear to $\vec{a}$ if it has the same direction and a magnitude greater than 10.

Since $|\vec{b}| > 10$, $k(5) > 10$, and $k > 2$.

Using $k = 3$ produces $\vec{b}$ such that $|\vec{b}| = 15$ with a bearing of 315°.

Therefore, $\vec{b} = 3\vec{a}$.

Collinear Vectors

Example

Vector $\vec{a}$ has a magnitude of 5, with a bearing of 315°. Describe a vector $\vec{b}$ that is collinear to $\vec{a}$ if it has the opposite direction and a magnitude smaller than 4.

Since $|\vec{b}| < 4$, $-4 < k(5) < 0$, and $-\frac{4}{5} < k < 0$.

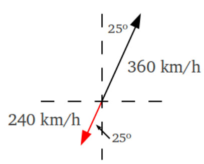
Using $k = -\frac{3}{5}$ produces $\vec{b}$ such that $|\vec{b}| = 3$ with a bearing of 135°.

Therefore, $\vec{b} = -\frac{3}{5}\vec{a}$.

Applications

Example

An airplane is flying with a velocity, $\vec{v}$, of 360 km/h N25°E. Draw a sketch of $-\frac{2}{3}\vec{v}$ and state its magnitude and direction.



The magnitude is 240 km/h, and the direction is S25°W.

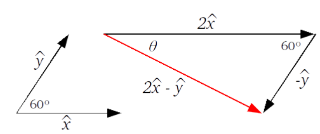
Applications

Example

The angle between $\hat{x}$ and $\hat{y}$ is 60°. Calculate the magnitude, and direction, of $2\hat{x} - \hat{y}$.

Recall that $\hat{x}$ and $\hat{y}$ are unit vectors, so $|\hat{x}| = |\hat{y}| = 1$.

To find the magnitude and direction of $2\hat{x} - \hat{y}$, use the following diagrams.



Applications

Use the cosine law to find the magnitude.

$$\begin{aligned} |2\hat{x} - \hat{y}| &= \sqrt{|2\hat{x}|^2 + |-\hat{y}|^2 - 2 \cdot |2\hat{x}| \cdot |-\hat{y}| \cdot \cos(60^\circ)} \\ &= \sqrt{2^2 + 1^2 - 2 \cdot 2 \cdot 1 \cdot \frac{1}{2}} \\ &= \sqrt{3} \end{aligned}$$

Use the sine law to find the angle, θ, relative to $\hat{x}$.

$$\begin{aligned} \frac{\sin\theta}{|-\vec{y}|} &= \frac{\sin 60^\circ}{|2\vec{x} - \vec{y}|} \\ \theta &= \sin^{-1}\left(\frac{1 \cdot \sin 60^\circ}{\sqrt{3}}\right) \\ \theta &= 30^\circ \end{aligned}$$

Multiplying a Vector By a Scalar

Distributive Property of Scalar Multiplication

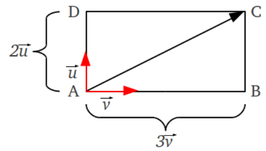
Given vectors $\vec{u}$ and $\vec{v}$ and scalar k, then $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$.

Recall that multiplication is simply repeated addition.

$$\begin{aligned} k(\vec{u} + \vec{v}) &= \underbrace{(\vec{u} + \vec{v}) + (\vec{u} + \vec{v}) + \ldots + (\vec{u} + \vec{v})}_{k \text{ times}} \\ &= \underbrace{\vec{u} + \vec{u} + \ldots + \vec{u}}_{k \text{ times}} + \underbrace{\vec{v} + \vec{v} + \ldots + \vec{v}}_{k \text{ times}} \\ &= k\vec{u} + k\vec{v} \end{aligned}$$

Linear Combinations of Vectors

Scalar multiplication can be combined with addition and subtraction. For example, the diagram below shows a rectangle where $\vec{AC} = 2\vec{u} + 3\vec{v}$.



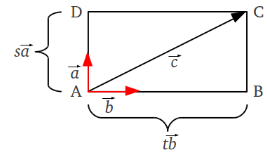
$\vec{AC}$ is a *linear combination* of vectors $\vec{u}$ and $\vec{v}$.

Linear Combinations of Vectors

Linear Combinations

Any vector $\vec{c}$ in a plane can be expressed as a distinct linear combination of two non-collinear vectors $\vec{a}$ and $\vec{b}$.

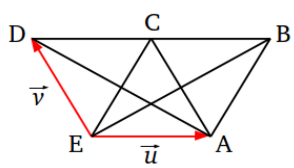
For unique scalars s and t, $\vec{c} = s\vec{a} + t\vec{b}$.



Linear Combinations of Vectors

Example

In the diagram below, triangles DEC, ECA and CAB are equilateral. Express $\vec{EC}$ as linear combinations of $\vec{u}$ and $\vec{v}$.

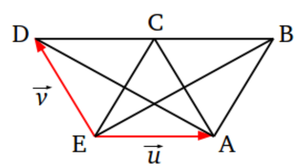


$\vec{EC} = \vec{EA} + \vec{AC} = \vec{EA} + \vec{ED} = \vec{u} + \vec{v}$

Linear Combinations of Vectors

Example

In the diagram below, triangles DEC, ECA and CAB are equilateral. Express $\vec{AD}$ as linear combinations of $\vec{u}$ and $\vec{v}$.

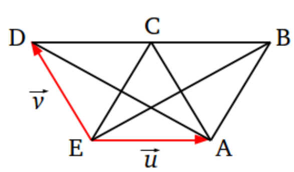


$\vec{AD} = \vec{AE} + \vec{ED} = \vec{ED} - \vec{EA} = \vec{v} - \vec{u}$

Linear Combinations of Vectors

Example

In the diagram below, triangles DEC, ECA and CAB are equilateral. Express $\vec{EB}$ as linear combinations of $\vec{u}$ and $\vec{v}$.



$\vec{EB} = \vec{ED} + \vec{DC} + \vec{CB} = \vec{ED} + \vec{EA} + \vec{EA} = \vec{v} + 2\vec{u}$

Linear Combinations of Vectors

Example

If $\vec{u} = 3\vec{x} - 2\vec{y}$ and $\vec{v} = 2\vec{x} - 5\vec{y}$, express $3\vec{u} - 4\vec{v}$ in terms of $\vec{x}$ and $\vec{y}$.

Substitute the definitions of $\vec{u}$ and $\vec{v}$ into the expression $3\vec{u} - 4\vec{v}$.

$$\begin{aligned} 3\vec{u} - 4\vec{v} &= 3(3\vec{x} - 2\vec{y}) - 4(2\vec{x} - 5\vec{y}) \\ &= 9\vec{x} - 6\vec{y} - 8\vec{x} + 20\vec{y} \\ &= \vec{x} + 14\vec{y} \end{aligned}$$

Questions?