MCV4U: Calculus & Vectors

Logarithmic Differentiation

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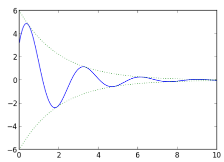

Derivatives Involving Exponential/Logarithmic Functions

Recap

Determine the derivative of $f(x) = 3^x \log_4(5x^2 - 3x)$.

Use the product and chain rules here.

$$f'(x) = 3^x \ln 3 \cdot \log_4(5x^2 - 3x) + 3^x \cdot \frac{10x-3}{(5x^2-3x)\ln 4}$$

Logarithmic Differentiation

We have covered several derivative rules so far (e.g. power rule, product rule, chain rule), as well as implicit differentiation.

Logarithmic differentiation is a technique that introduces logarithms into a function in order to rewrite it in a differentiable way.

This is usually done by using the property $\log_b p^q = q \log_b p$, "bringing down" the exponent as a product, then using implicit differentiation to find the derivative.

Many functions *require* the use of logarithmic differentiation to find their derivatives.

Since the natural logarithm is generally easier to work with than a general logarithm, most solutions involve ln.

Logarithmic Differentiation

Example

Determine the derivative of $y = x^x$.

In this case, y is neither a polynomial (the exponent is not a constant) nor an exponential function (the base is not a constant), so neither rule applies.

By taking the natural logarithm of both sides, however, we are able to "pull down" the exponent as a constant multiple using logarithm rules.

$$\begin{aligned} \ln y &= \ln x^x \\ \ln y &= x \ln x \end{aligned}$$

Logarithmic Differentiation

Now we can use implicit differentiation. Be sure to use the product rule on the right hand side.

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \ln x + x \cdot \frac{1}{x} \\ &= \ln x + 1 \\ \frac{dy}{dx} &= y(\ln x + 1) \\ &= x^x(\ln x + 1) \end{aligned}$$

Logarithmic Differentiation

Example

Determine the derivative of $y = (\sin x)^{2x}$.

As before, take the natural logarithm of both sides and bring down the exponent.

$$\begin{aligned} \ln y &= 2x \ln(\sin x) \\ \frac{1}{y}\frac{dy}{dx} &= 2\ln(\sin x) + 2x \cdot \frac{\cos x}{\sin x} \\ \frac{dy}{dx} &= y\left(\cdot 2\ln(\sin x) + 2x \cdot \frac{\cos x}{\sin x}\right) \\ &= (\sin x)^{2x} \cdot \left(2\ln(\sin x) + 2x \cdot \frac{\cos x}{\sin x}\right) \end{aligned}$$

Logarithmic Differentiation

Logarithmic differentiation can be used with *any* function, but would not generally make sense for simple functions (e.g. polynomials), since it introuduces both implicit differentiation and the chain rule.

On the other hand, logarithmic differentiation can be used as an alternative when functions are significantly more complex.

In most cases, logarithmic differentiation will involve logarithm laws to either simplify the derivative, or to make the process more straightforward (though not necessarily shorter).

Logarithmic Differentiation

Example

Determine the derivative of $y = \dfrac{x^3}{\sqrt{x^2+1}(1-2x)}$.

We can, of course, use the product, quotient and chain rules to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{3x^2\sqrt{x^2+1}(1-2x) - x^3\left(-2\sqrt{x^2+1} + (1-2x)\left(\frac{2x}{2\sqrt{x^2+1}}\right)\right)}{(x^2+1)(1-2x)^2}$$

This is pretty intimidating, and difficult to simplify. Instead, repeat the question using logarithmic differentiation.

Logarithmic Differentiation

Begin by taking the natural logarithm of both sides.

$$\begin{aligned} y &= \frac{x^3}{\sqrt{x^2+1}(1-2x)} \\ \ln y &= \ln\left(\frac{x^3}{\sqrt{x^2+1}(1-2x)}\right) \end{aligned}$$

Now, use the properties $\log_b(pq) = \log_b p + \log_b q$ and $\log_b\left(\frac{p}{q}\right) = \log_b p - \log_b q$.

$$\begin{aligned} \ln y &= \ln x^3 - \ln(\sqrt{x^2+1}(1-2x)) \\ &= \ln x^3 - \ln\sqrt{x^2+1} - \ln(1-2x) \end{aligned}$$

Logarithmic Differentiation

Now use implicit differentiation, along with the chain rule.

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{3x^2}{x^3} - \frac{\frac{2x}{2\sqrt{x^2+1}}}{\sqrt{x^2+1}} - \frac{-2}{1-2x} \\ &= \frac{3}{x} - \frac{2x}{2(x^2+1)} + \frac{2}{1-2x} \\ \frac{dy}{dx} &= y\left(\frac{3}{x} - \frac{2x}{2(x^2+1)} + \frac{2}{1-2x}\right) \\ &= \frac{x^3}{\sqrt{x^2+1}(1-2x)}\left(\frac{3}{x} - \frac{2x}{2(x^2+1)} + \frac{2}{1-2x}\right) \end{aligned}$$

While the answer itself is not a whole lot "simpler", the process is arguably easier – finding the derivative of the three terms involving ln uses only the chain rule.

Questions?