MCV4U: Calculus & Vectors

Equations of Lines In Two-Space

Part 2: Symmetric and Scalar Equations

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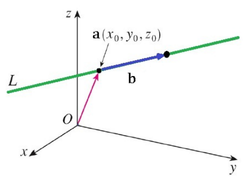

Vector and Parametric Equations of a Line

Recap

Determine vector and parametric equations of the line parallel to $\vec{r} = (1, 4) + t(-2, 3)$ that passes through $(3, -5)$.

The direction vector for $\vec{r}$ is $(-2, 3)$.

Therefore, a vector equation of the new line is $\vec{r_2} = (3, -5) + s(-2, 3)$.

The corresponding parametric equations are $x = 3 - 2s$ and $y = -5 + 3s$.

Symmetric Equation of a Line

Parametric equations have the form $x = x_p + x_m t$ and $y = y_p + y_m t$, for some position vector $\vec{p} = (x_p, y_p)$ are direction vector $\vec{m} = (x_m, y_m)$.

Rearranging each equation, $t = \frac{x - x_p}{x_m}$ and $t = \frac{y - y_p}{y_m}$.

Since the parameter, t, is the same in both equations, we can equate the right hand sides.

Symmetric Equation of a Line In Two Space

The symmetric equation of a line in two-space is $\frac{x - x_p}{x_m} = \frac{y - y_p}{y_m}$, where $\vec{p} = (x_p, y_p)$ a position vector representing a point on the line and $\vec{m} = (x_m, y_m)$ is a direction vector for the line.

Symmetric Equation of a Line

Example

State the symmetric equation of the line defined by the parametric equations $x = -3 + 2t$ and $y = 5 - 4t$.

Since $t = \dfrac{x + 3}{2}$ and $t = \dfrac{y - 5}{-4}$, the equation is $\dfrac{x + 3}{2} = -\dfrac{y - 5}{4}$ or $\dfrac{x + 3}{2} = \dfrac{5 - y}{4}$.

Symmetric Equation of a Line

Example

State the symmetric equation of the line defined by the equation $\vec{r} = (4, -1) + s(3, 7)$.

A position vector is $\vec{p} = (4, -1)$ and a direction vector is $\vec{m} = (3, 7)$, so the symmetric equation is $\dfrac{x - 4}{3} = \dfrac{y + 1}{7}$.

Normal to a Line

Another method of defining a line is to use a *normal* vector, $\vec{n}$, which is perpendicular to the line.

Recall that perpendicular lines in two-space have negative reciprocal slopes.

Example

State a normal vector to $\vec{r} = (1, -2) + t(-3, 4)$.

Since $\vec{m} = (-3, 4)$, $\vec{n} = (4, 3)$.

Using the dot product to verify, $-3(4) + 4(3) = 0$.

Normal to a Line

Example

Represent the equation of the line with normal vector $\vec{n} = (2, 5)$ that passes through $P(-1, 3)$ using parametric, vector, and slope-intercept forms.

A direction vector for the line is perpendicular to $\vec{n}$, so $\vec{m} = (-5, 2)$.

The parametric equations of the line are $x = -1 - 5t$ and $y = 3 + 2t$.

The vector equation is $\vec{r} = (-1, 3) + t(-5, 2)$.

Normal to a Line

For slope-intercept form, solve for the y-intercept, b, in $y = mx + b$.

$$\begin{aligned} 3 &= -\tfrac{2}{5}(-1) + b \\ b &= \tfrac{13}{5} \\ y &= -\tfrac{2}{5}x + \tfrac{13}{5} \end{aligned}$$

Scalar Equation of a Line

Multiplying both sides of the equation by 5 and moving all terms to one side, we obtain the equation $2x + 5y - 13 = 0$.

The equation, now written in *scalar* form (or *standard* form), has the normal vector, $\vec{n}$, as the coefficients of x and y.

Scalar Equation of a Line In Two Space

The scalar equation of a line in two-space is $Ax + By + C = 0$, where $\vec{n} = (A, B)$ is a normal to the line.

Scalar Equation of a Line

Example

Determine the scalar equation of the line with normal vector $\vec{n} = (8, -3)$ that passes through $P(-2, 1)$.

Use $A = 8$ and $B = -3$ in the equation.

$$\begin{aligned} 8(-2) - 3(1) + C &= 0 \\ C &= 19 \\ 8x - 3y + 19 &= 0 \end{aligned}$$

Scalar Equation of a Line

Example

A line in two-space has scalar equation $2x - 5y - 15 = 0$. Determine the parametric equations of the line.

A normal to the line is $\vec{n} = (2, -5)$, so a direction vector for the line is $\vec{m} = (5, 2)$.

Substitute $x = 0$ (or any value) into the equation to find a point on the line.

$$\begin{aligned} 2(0) - 5y - 15 &= 0 \\ y &= -3 \end{aligned}$$

Thus, parametric equations for the line are $x = 5t$ and $y = -3 + 2t$.

Questions?