MCV4U: Calculus & Vectors

Equations of Lines In Two-Space

Part 1: Parametric and Vector Equations

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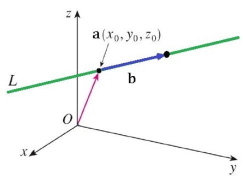
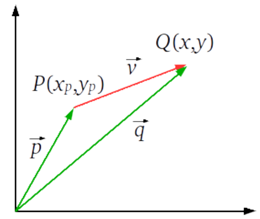


Parametric Equations of a Line

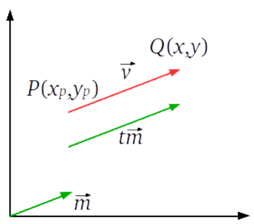
Imagine moving from some initial point on a line, $P(x_p, y_p)$, to a new point, $Q(x, y)$.



A vector representing the change in location is $\vec{v} = (x - x_p, y - y_p)$.

Parametric Equations of a Line

Another way to represent this is to use a scalar multiple of a *direction vector*, $\vec{m} = (x_m, y_m)$, that is parallel to $\vec{v}$.



A vector from P to Q is $\vec{v} = t\vec{m}$ or $\vec{v} = (tx_m, ty_m)$.

The direction vector is similar to slope, in that a slope of $\frac{a}{b}$ yields a direction vector of $\vec{m} = (b, a)$.

Parametric Equations of a Line

Equating the two representations of $\vec{v}$, we get

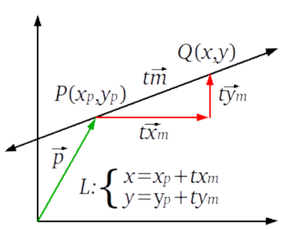
$$x - x_p = tx_m \qquad y - y_p = ty_m$$
$$x = x_p + tx_m \qquad y = y_p + ty_m$$

Since any scalar multiple of $\vec{v}$ creates a vector, a line can be represented as the set of all possible vectors that are scalar multiples of $\vec{v}$.

Parametric Equations of a Line

Parametric Equations of a Line In Two-Space

The *parametric equations* of a line in two-space are $x = x_p + tx_m$ and $y = y_p + ty_m$, where $\vec{p} = (x_p, y_p)$ is a position vector representing a point on the line, $\vec{m} = (x_m, y_m)$ is a direction vector for the line, and $t \in R$.



Parametric Equations of a Line

Example

Determine the parametric equations of the line that passes through $A(3, -5)$ with direction vector $(-2, 3)$.

The equations are $x = 3 - 2t$ and $y = -5 + 3t$.

Example

Does the point $B(-13, -1)$ lie on the line with parametric equations $x = 2 - 5t$ and $y = -4 + t$?

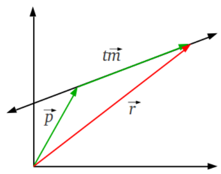
Substitute the coordinates for point B into the equations.

$$-13 = 2 - 5t \qquad -1 = -4 + t$$
$$t = 3 \qquad t = 3$$

Since the parameter t is the same in both equations, point B is on the line.

Vector Equation of a Line

A line can also be represented using a *vector equation*.



By starting at point $P(x_p, y_p)$ and moving in the same or opposite direction of $\vec{m}$, we can reach any point on the line.

This point will have a position vector equal to the resultant of $\vec{p} + t\vec{m}$, where t is some real value.

Vector Equation of a Line

Vector Equation of a Line In Two-Space

The vector equation of a line in two-space is $\vec{r} = (x_p, y_p) + t(x_m, y_m)$, where $\vec{p} = (x_p, y_p)$ is a position vector of a point on the line, $\vec{m} = (x_m, y_m)$ is a direction vector for the line, and $t \in R$.

Vector Equation of a Line

Example

Determine the vector equation of the line that passes through $A(3, -1)$ and $B(6, 8)$.

A direction vector is $\vec{m} = (3, 9)$, but any scalar multiple will do, such as $\vec{m} = (1, 3)$.

Using point A, a possible equation for the line is $\vec{r} = (3, -1) + t(1, 3)$.

Using point B, another possible equation is $\vec{r} = (6, 8) + t(1, 3)$.

Vector Equation of a Line

Example

Represent the line perpendicular to $\vec{r} = (3, -4) + t(-5, 1)$ and passing through $P(12, 3)$ using both parametric and vector equations.

A direction vector perpendicular to $(-5, 1)$ is $(1, 5)$.

Therefore, a vector equation of the line through $P(12, 3)$ is $\vec{r} = (12, 3) + t(1, 5)$.

The corresponding parametric equations of the line are $x = 12 + t$ and $y = 3 + 5t$.

Vector Equation of a Line

Example

Determine whether the two lines $\vec{r_1} = (5, -2) + s(6, -3)$ and $\vec{r_2} = (-3, 4) + t(-4, 2)$ are coincident.

Direction vectors for $\vec{r_1}$ and $\vec{r_2}$ are $\vec{m_1} = (6, -3)$ and $\vec{m_2} = (-4, 2)$.

Since $(6, -3) = -\frac{3}{2}(-4, 2)$, $\vec{r_1}$ and $\vec{r_2}$ are parallel.

To see if the lines are coincident, see if a point on $\vec{r_1}$ is also on $\vec{r_2}$. Try $(5, -2) = (-3, 4) + t(-4, 2)$.

$$5 = -3 - 4t \qquad -2 = 4 + 2t$$
$$t = -2 \qquad t = -3$$

Since we obtain different values for t, the lines are not coincident.

Questions?