MCV4U: Calculus & Vectors

Limits In Indeterminate Form

J. Garvin

Limits

Recap

Use limit properties to evaluate $\lim_{x\to5}(3x^2-7)$.

$$\begin{aligned}\lim_{x\to5}(3x^2-7)&=\lim_{x\to5}3x^2-\lim_{x\to5}7\\&=3\lim_{x\to5}x^2-\lim_{x\to5}7\\&=3(\lim_{x\to5}x)^2-\lim_{x\to5}7\\&=3(5)^2-7\\&=68\end{aligned}$$

Limits

What happens when we try to evaluate a limit like $\lim_{x\to3}\frac{x^2-9}{x-3}$?

At some point, the denominator becomes $\lim_{x\to3}x-\lim_{x\to3}3=3-3=0$, which suggests that the answer is undefined.

However, the numerator becomes $(\lim_{x\to3}x)^2-\lim_{x\to3}9=9-9=0$ as well.

This expression, $\frac{0}{0}$, is one of several *indeterminate forms*. That is, it is not immediately evident what the limit should be as $x\to3$.

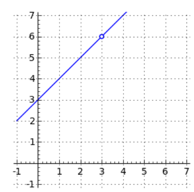
Limits

Indeterminate Form

A limit is in indeterminate form if it can be written in any of the following ways:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^0,\ 1^\infty,\ \infty^0$$

Indeed, a graph suggests that $\lim_{x\to3}\frac{x^2-9}{x-3}$ should be 6.



Limits in Indeterminate Form

Example

Determine $\lim_{x\to3}\frac{x^2-9}{x-3}$ algebraically.

Note that the numerator is a difference of squares, and can be factored.

$$\begin{aligned}\lim_{x\to3}\frac{x^2-9}{x-3}&=\lim_{x\to3}\frac{(x-3)(x+3)}{x-3}\\&=\lim_{x\to3}(x+3),x\neq3\\&=\lim_{x\to3}x+\lim_{x\to3}3\\&=3+3\\&=6\end{aligned}$$

Remember that a limit is a value that is *approached*.

Limits in Indeterminate Form

Example

Determine $\lim_{x\to-2}\frac{x^3+8}{x+2}$ algebraically.

The numerator is a sum of cubes, and can be factored.

$$\begin{aligned}\lim_{x\to-2}\frac{x^3+8}{x+2}&=\lim_{x\to-2}\frac{(x+2)(x^2-2x+4)}{x+2}\\&=\lim_{x\to-2}(x^2-2x+4),x\neq-2\\&=(\lim_{x\to-2}x)^2-2\lim_{x\to-2}x+\lim_{x\to-2}4\\&=(-2)^2-2(-2)+4\\&=12\end{aligned}$$

Limits in Indeterminate Form

Example

Determine $\lim_{x\to 1} \frac{x^3+6x^2+3x-10}{x-1}$ algebraically.

$(x-1)$ is a factor, since $1^3+6(1)^2+3(1)-10=0$.

Use division to determine the resulting quadratic expression.

	1	6	3	−10
1		1	7	10
	1	7	10	0

Therefore, $x^3+6x^2+3x-10=(x-1)(x^2+7x+10)$.

Limits in Indeterminate Form

$$\begin{aligned}\lim_{x\to 1} \frac{x^3+6x^2+3x-10}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x^2+7x+10)}{x-1}\\ &= \lim_{x\to 1}(x^2+7x+10), x\neq 1\\ &= (\lim_{x\to 1} x)^2+7\lim_{x\to 1} x+\lim_{x\to 1} 10\\ &= (1)^2+7(1)+10\\ &= 18\end{aligned}$$

Limits in Indeterminate Form

Example

Determine $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ algebraically.

Rationalizing a limit often eliminates indeterminate forms.

$$\begin{aligned}\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \times \frac{\sqrt{x}+3}{\sqrt{x}+3}\\ &= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}\\ &= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}\\ &= \frac{\lim_{x\to 9} 1}{\sqrt{\lim_{x\to 9} x+\lim_{x\to 9} 3}}\\ &= \frac{1}{6}\end{aligned}$$

Limits in Indeterminate Form

Example

Determine $\lim_{x\to 0} \frac{\sqrt[3]{x+8}-2}{x}$ algebraically.

Rationalizing will not work here. Instead, we can do an algebraic substitution, letting $u=\sqrt[3]{x+8}$.

Therefore, $x=u^3-8$, which is a difference of cubes.

Note that as $x\to 0$, then $u\to 2$.

This means we can rewrite the limit in terms of u instead.

Limits in Indeterminate Form

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt[3]{x+8}-2}{x} &= \lim_{u\to 2} \frac{u-2}{u^3-8}\\ &= \lim_{u\to 2} \frac{u-2}{(u-2)(u^2+2u+4)}\\ &= \lim_{u\to 2} \frac{1}{u^2+2u+4}\\ &= \frac{\lim_{u\to 2} 1}{\lim_{u\to 2} u^2+2\lim_{u\to 2} u+\lim_{u\to 2} 4}\\ &= \frac{1}{12}\end{aligned}$$

Questions?