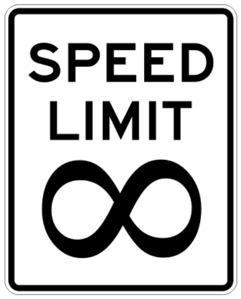
MCV4U: Calculus & Vectors

The Derivative of a Function

J. Garvin



Slope of a Tangent

Recap

Determine the slope of the tangent to $f(x) = 5x^2$ when $x = -2$.

Use the difference quotient with $x = -2$ and $f(-2) = 20$.

$$\begin{aligned} m_{\text{tangent}} &= \frac{5(-2+h)^2 - 20}{h} \\ &= \frac{5(4 - 4h + h^2) - 20}{h} \\ &= \frac{-20h + 5h^2}{h} \\ &= -20 + 5h \end{aligned}$$

As $h \to 0$, $-20 + 5h \to -20$. Therefore, the slope of the tangent to $f(x) = 5x^2$ at $x = -2$ is -20.

The Derivative of a Function

In the previous example, we let $h \to 0$ to create an infinitesimally small interval for our secant, essentially creating a tangent at $x = -2$.

We recognize this as the *limit* of the difference quotient as $h \to 0$.

This gives us a more formal definition using limits.

Limit Definition of the Derivative

For a function $y = f(x)$, the *derivative* may be expressed as:

- $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ in Lagrange notation, or
- $\frac{dy}{dx} = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ in Leibniz notation

The Derivative of a Function

The derivative is an expression that can be used to evaluate the instantaneous rate of change at a given point on a curve, or the slope of the tangent at that point.

The process of determining the derivative of a function is called *differentiation*.

If the derivative can be found at a given point, then the function is *differentiable* at that point. It is not always possible to determine the derivative of a function for all points on its domain.

Using this limit definition of the derivative is sometimes referred to as *differentiating using first principles*.

The Derivative of a Function

Example

Determine the derivative of $f(x) = 3x^2 + 5$ at $x = 2$.

We can use the limit definition of the derivative to verify this in one of two ways:

- Substitute $x = 2$ into the expression, and find the derivative for that point alone, or
- Determine a general expression for the derivative, and evaluate it after-the-fact for $x = 2$

If a question asks for a one-off answer, the first method is fine. For applications that require further analysis, or ask for the derivative at multiple points, the second is better.

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Using the first method, substitute $x = 2$ and $f(2) = 17$ into the expression.

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{3(2+h)^2 + 5 - 17}{h} \\ &= \lim_{h \to 0} \frac{3(4 + 4h + h^2) - 12}{h} \\ &= \lim_{h \to 0} \frac{12h + 3h^2}{h} \\ &= \lim_{h \to 0} \frac{h(12 + 3h)}{h} \\ &= \lim_{h \to 0} (12 + 3h) \\ &= \lim_{h \to 0} 12 + 3 \lim_{h \to 0} h \\ &= 12 \end{aligned}$$

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Using the second method, expand and simplify.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[3(x+h)^2+5]-[3x^2+5]}{h} \\ &= \lim_{h\to 0} \frac{3(x^2+2xh+h^2)+5-3x^2-5}{h} \\ &= \lim_{h\to 0} \frac{6xh+3h^2}{h} \\ &= \lim_{h\to 0} \frac{h(6x+3h)}{h} \\ &= \lim_{h\to 0}(6x+3h) \\ &= 6\lim_{h\to 0} x + 3\lim_{h\to 0} h \\ &= 6x \end{aligned}$$

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Now that we have a general expression for $f'(x)$, we can use it to evaluate the rate of change when $x = 2$.

$$\begin{aligned} f'(2) &= 6(2) \\ &= 12 \end{aligned}$$

Both methods found an instantaneous rate of change of 12 when $x = 2$.

If we needed to evaluate the instantaneous rate of change at other points on the curve, say at $x = 10$, the second method tells us that $f'(10) = 6(10) = 60$.

Had we used the first method, we would have to calculate $f'(10)$ from scratch.

The Derivative of a Function

Example

Determine the instantaneous rates of change of $y = \sqrt{x}$ when $x = 4$ and $x = 9$.

We need to find both $\left.\frac{dy}{dx}\right|_{x=4}$ and $\left.\frac{dy}{dx}\right|_{x=9}$.

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \end{aligned}$$

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Recall that radicals can be written in exponential form.

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{1}{(x+h)^{\frac{1}{2}} + x^{\frac{1}{2}}} \\ &= \frac{\lim_{h\to 0} 1}{\left(\lim_{h\to 0} x + \lim_{h\to 0} h\right)^{\frac{1}{2}} + \left(\lim_{h\to 0} x\right)^{\frac{1}{2}}} \\ &= \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{2}}} \\ &= \frac{1}{2\sqrt{x}} \text{ or } \frac{1}{2}x^{-\frac{1}{2}} \end{aligned}$$

Therefore, $\left.\frac{dy}{dx}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ and $\left.\frac{dy}{dx}\right|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

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Example

Determine the equation of the tangent to $f(x) = \frac{1}{x}$ at $x = 5$.

The slope of the tangent at $x = 5$ is given by $f'(5)$.

$$\begin{aligned} f'(5) &= \lim_{h\to 0} \frac{\frac{1}{5+h} - \frac{1}{5}}{h} \\ &= \lim_{h\to 0} \frac{5-(5+h)}{5h(5+h)} \\ &= \lim_{h\to 0} \frac{-1}{5(5+h)} \\ &= \frac{\lim_{h\to 0}(-1)}{\lim_{h\to 0} 25 + \lim_{h\to 0} h} \\ &= -\frac{1}{25} \end{aligned}$$

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When $x = 5$, $f(5) = \frac{1}{5}$, so the point of tangency is $\left(5, \frac{1}{5}\right)$.

Use the slope and point to find the equation of the tangent.

$$\begin{aligned} y &= -\tfrac{1}{25}(x-5) + \tfrac{1}{5} \\ &= -\tfrac{1}{25}x + \tfrac{1}{5} + \tfrac{1}{5} \\ &= -\tfrac{1}{25}x + \tfrac{2}{5} \end{aligned}$$

Therefore, the tangent to $f(x) = \frac{1}{x}$ at $x = 5$ has the equation $y = -\frac{1}{25}x + \frac{2}{5}$.

Questions?