MCV4U: Calculus & Vectors

Intersections of Lines

Part 2: Lines in R^3

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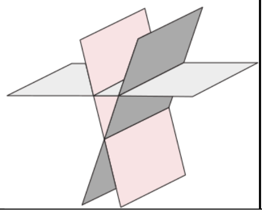
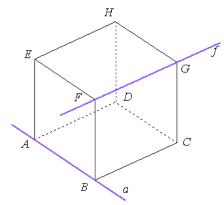


Intersections of Lines In Three-Space

Recall that in two-space, lines may either be parallel, coincident, or intersecting in a single point.

These options still exist in three-space, but there is an additional option to consider.

In three-space, two non-parallel lines may not intersect. Such lines are called *skew* lines.



Intersections of Lines In Three-Space

A general strategy for determining a point of intersection of two lines in three-space is as follows:

1. Check the direction vectors of the lines. If they are scalar multiples, the lines are either parallel and distinct, or coincident.
   - If a known point on the first line is also on the second line, the lines are coincident, with infinite solutions.
   - If the known point is not on the second line, the lines are parallel and distinct, with no solution.
2. If the direction vectors are not scalar multiples, the lines either intersect in a single point or are skew. Solve for the parameters (s and t).
   - If the parameters satisfy all three equations, there is a single point of intersection.
   - If the parameters fail to satisfy all three equations, the lines are skew, with no solution.

Intersections of Lines In Three-Space

Example

Determine if the lines $\vec{L_1} = (1,2,3) + s(5,-2,3)$ and $L_2 : \frac{x-1}{5} = \frac{y+3}{-2} = \frac{z+1}{3}$ intersect, or if they are parallel or skew.

First, check to see if the two lines are parallel or coincident.

Direction vectors are $\vec{m_1} = (5,-2,3)$ and $\vec{m_2} = (5,-2,3)$, which are collinear. Therefore, the lines are either parallel or coincident.

Test if point $(1,2,3)$ from $\vec{L_1}$ is on $\vec{L_2}$:

$$\frac{1-1}{5} = 0 \qquad \frac{2+3}{-2} = -\frac{5}{2} \qquad \frac{3+1}{3} = \frac{4}{3}$$

Since the values are not the same in the symmetric equation, $(1,2,3)$ is not on $\vec{L_2}$. The lines are parallel and distinct.

Intersections of Lines In Three-Space

Example

Determine the point of intersection of the lines $\vec{L_1} = (3,-7,5) + s(1,-2,4)$ and $L_2 : \frac{x+7}{3} = \frac{y+8}{1} = \frac{z-4}{-1}$, if one exists.

First, check to see if the two lines are parallel or coincident.

Direction vectors are $\vec{m_1} = (1,-2,4)$ and $\vec{m_2} = (3,1,-1)$, which are not collinear. Therefore, the lines either have a single point of intersection or they are skew.

Write the parametric equations for the two lines.

$$L_1 : x = 3 + s \qquad L_2 : x = -7 + 3t$$
$$y = -7 - 2s \qquad y = -8 + t$$
$$z = 5 + 4s \qquad z = 4 - t$$

Intersections of Lines In Three-Space

If there is a single point of intersection, the values of x, y and z will be equal.

$$\begin{aligned} 3 + s &= -7 + 3t \\ -7 - 2s &= -8 + t \\ 5 + 4s &= 4 - t \end{aligned}$$

Add the second and third equations to eliminate t.

$$\begin{aligned} -2 + 2s &= -4 \\ s &= -1 \end{aligned}$$

Intersections of Lines In Three-Space

Substitute $s = -1$ into the second equation.

$$-7 - 2(-1) = -8 + t$$
$$t = 3$$

Since we used the second and third equations to solve for s and t, we need to verify the values work in the first equation.

$$LHS = 3 + s = 3 + (-1) = 2 \qquad RHS = -7 + 3t = -7 + 3(3) = 2$$

Since the values are the same, the linear system is consistent and there is a single point of intersection.

Intersections of Lines In Three-Space

To find the point of intersection, substitute either $s = -1$ or $t = 3$ into the parametric equations of either L_1 or L_2.

Using L_1:

$$x = 3 + s = 3 + (-1) = 2 \qquad y = -7 - 2s = -7 - 2(-1) = -5 \qquad z = 5 + 4s = 5 + 4(-1) = 1$$

Therefore, the point of intersection is $(2, -5, 1)$.

Intersections of Lines In Three-Space

Example

Determine the point of intersection of the lines $\vec{L_1} = (4, -5, 1) + s(2, 4, 3)$ and $\vec{L_2} = (2, -1, 0) + t(1, 3, 2)$, if one exists.

Since the direction vectors are not scalar multiples, the lines either have a single point of intersection or they are skew.

Parametric equations for the two lines are:

$$L_1 : x = 4 + 2s,\quad y = -5 + 4s,\quad z = 1 + 3s \qquad L_2 : x = 2 + t,\quad y = -1 + 3t,\quad z = 2t$$

Intersections of Lines In Three-Space

Equate the parametric equations:

$$4 + 2s = 2 + t$$
$$-5 + 4s = -1 + 3t$$
$$1 + 3s = 2t$$

Multiply the first equation by 2 and subtract the second to eliminate s.

$$8 + 4s = 4 + 2t$$
$$-5 + 4s = -1 + 3t$$
$$13 = 5 - t$$
$$t = -8$$

Using $t = -8$ in the first equation yields $s = -5$.

Intersections of Lines In Three-Space

If the lines intersect, the parameters will work in all three equations.

In the third equation, however, using $t = -8$ and $s = -5$ produces the equation $14 = -16$, which is not true.

Therefore, the lines are skew and do not intersect.

Questions?