MCV4U: Calculus & Vectors

Intersections of Three Planes

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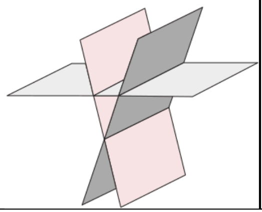


Intersections of Three Planes

There are many more ways in which three planes may intersect (or not) than two planes.

First consider the cases where all three normals are collinear.

- All three planes are parallel and distinct (inconsistent)
- Two planes are coincident, and the third is parallel (inconsistent)
- All three planes are coincident (infinite solutions)

Intersections of Three Planes

Example

Determine any points of intersection of the planes $\pi_1 : 2x - y + 5z + 4 = 0$, $\pi_2 : 4x - 2y + 10z + 15 = 0$ and $\pi_3 : -6x + 3y - 15z + 7 = 0$.

The three normals are $\vec{n_1} = (2, -1, 5)$, $\vec{n_2} = (4, -2, 10)$ and $\vec{n_3} = (-6, 3, -15)$.

$\vec{n_2} = 2\vec{n_1}$, but the equation for π_2 is not twice that of π_1.

Similarly, $\vec{n_3} = -3\vec{n_1}$, but the equation for π_3 is not triple that of π_1.

Therefore, the plans are parallel and distinct, and there are no points of intersection.

Intersections of Three Planes

Example

Determine any points of intersection of the planes $\pi_1 : 3x - 2y + 1z + 5 = 0$, $\pi_2 : 6x - 4y + 2z + 10 = 0$ and $\pi_3 : 15x - 10y + 5z + 25 = 0$.

The three normals are $\vec{n_1} = (3, -2, 1)$, $\vec{n_2} = (6, -4, 2)$ and $\vec{n_3} = (15, -10, 5)$.

The equations for π_2 and π_3 are multiples of that of π_1 (by 2 and by 5).

Therefore, the planes are all coincident.

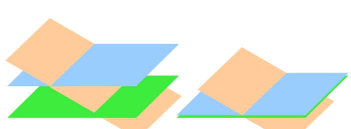
As before, let s and t be parameters and set $y = s$ and $z = t$. Then $x = \frac{2}{3}s + \frac{1}{3}t - 5$.

These form the parametric equations of the plane that contains all solutions.

Intersections of Three Planes

Next, consider the cases where only two normals are collinear.

- Two planes are coincident, and the third cuts the others (intersection is a line)
- Two planes are parallel, and the third cuts the others (inconsistent)

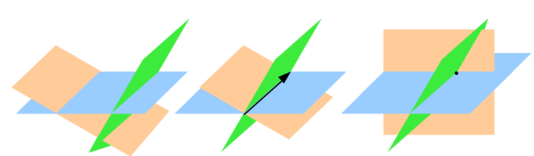


In the case of the first scenario, solve as earlier using the intersection of two planes.

Intersections of Three Planes

Finally, consider the cases where none of the normals are collinear.

- Normals are coplanar, planes intersect in pairs (inconsistent)
- Normals are coplanar, planes intersect each other (intersection is a line)
- Normals not coplanar (intersection is a point)



Intersections of Three Planes

Example

Determine any points of intersection of the planes $\pi_1 : x - y + z + 2 = 0$, $\pi_2 : 2x - y - 2z + 9 = 0$ and $\pi_3 : 3x + y - z + 2 = 0$.

By inspection, none of the normals are collinear.

Check if the normals are coplanar using the triple scalar product.

$$\begin{aligned}(\vec{n_1} \times \vec{n_2}) \cdot \vec{n_3} &= ((1, -1, 1) \times (2, -1, -2)) \cdot (3, 1, -1) \\ &= (3, 4, 1) \cdot (3, 1, -1) \\ &= 12\end{aligned}$$

Since the triple scalar product is non-zero, the normals are not coplanar and the planes intersect at a single point.

Intersections of Three Planes

We need to solve a linear system of three equations with three unknowns.

$$\begin{aligned} x - y + z + 2 &= 0 \\ 2x - y - 2z + 9 &= 0 \\ 3x + y - z + 2 &= 0 \end{aligned}$$

Choose one variable that is easy to eliminate in two different pairs of equations, such as y.

Using equations 1 and 2, eliminate y.

$$\begin{aligned} x - y + z + 2 &= 0 \\ - \quad 2x - y - 2z + 9 &= 0 \\ -x + 3z - 7 &= 0 \end{aligned}$$

Intersections of Three Planes

Using equations 2 and 3, eliminate y.

$$\begin{aligned} 2x - y - 2z + 9 &= 0 \\ + \quad 3x + y - z + 2 &= 0 \\ 5x - 3z + 11 &= 0 \end{aligned}$$

Now we have a system of two equations with two unknowns, and we can solve for x.

$$\begin{aligned} -x + 3z - 7 &= 0 \\ + \quad 5x - 3z + 11 &= 0 \\ 4x + 4 &= 0 \\ x &= -1 \end{aligned}$$

Intersections of Three Planes

Solve for z.

$$\begin{aligned} 5(-1) - 3z + 11 &= 0 \\ z &= 2 \end{aligned}$$

Use the equation of any of the three planes to solve for y.

$$\begin{aligned} -1 - y + 2 + 2 &= 0 \\ y &= 3 \end{aligned}$$

The point of intersection is at $(-1, 3, 2)$.

Intersections of Three Planes

Example

Determine any points of intersection of the planes $\pi_1 : x - 5y + 2z - 10 = 0$, $\pi_2 : x + 7y - 2z + 6 = 0$ and $\pi_3 : 8x + 5y + z - 20 = 0$.

Again, none of the normals are collinear.

Check if the normals are coplanar using the triple scalar product.

$$\begin{aligned}(\vec{n_1} \times \vec{n_2}) \cdot \vec{n_3} &= ((1, -5, 2) \times (1, 7, -2)) \cdot (8, 5, 1) \\ &= (-4, 4, 12) \cdot (8, 5, 1) \\ &= 0\end{aligned}$$

Since the triple scalar product is zero, the normals are coplanar and the planes either intersect in a line or in pairs.

Intersections of Three Planes

Set up a linear system of three equations with three unknowns.

$$\begin{aligned} x - 5y + 2z - 10 &= 0 \\ x + 7y - 2z + 6 &= 0 \\ 8x + 5y + z - 20 &= 0 \end{aligned}$$

Using equations 1 and 2, eliminate z.

$$\begin{aligned} x - 5y + 2z - 10 &= 0 \\ + \quad x + 7y - 2z + 6 &= 0 \\ 2x + 2y - 4 &= 0 \end{aligned}$$

Intersections of Three Planes

Using equations 2 and 3, eliminate z.

$$\begin{aligned} x + 7y - 2z + 6 &= 0 \\ + \quad 16x + 10y + 2z - 40 &= 0 \\ 17x + 17y - 34 &= 0 \end{aligned}$$

Now we have a system of two equations with two unknowns.

$$\begin{aligned} 2x + 2y - 4 &= 0 \\ 17x + 17y - 34 &= 0 \end{aligned}$$

Intersections of Three Planes

Multiplying the first equation by 17 and the second by 2, we obtain the following system.

$$\begin{aligned} 34x + 34y - 68 &= 0 \\ 34x + 34y - 68 &= 0 \end{aligned}$$

This system is true for any values of x and y, so the planes intersect in a line. Let $x = t$ and solve for y and z.

$$\begin{aligned} 2t + 2y - 4 &= 0 \\ y &= 2 - t \\ t + 7(2 - t) - 2z + 6 &= 0 \\ z &= 10 - 3t \end{aligned}$$

The planes intersect in a line with parametric equations $x = t$, $y = 2 - t$ and $z = 10 - 3t$.

Questions?