MCV4U: Calculus & Vectors

Intersection of a Line and a Plane

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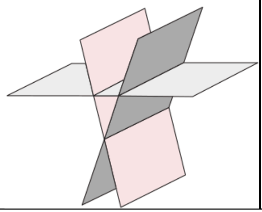
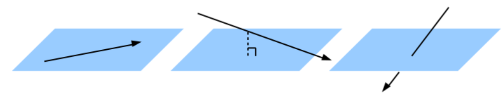


Intersection of a Line and a Plane

A line and a plane may or may not intersect.

There are three possibilities:

- The line is contained in the plane (infinite intersections)
- The line is parallel to the plane (no intersections)
- The line intersects the plane (one point of intersection)



Intersection of a Line and a Plane

If we are given the scalar equation of a plane, we are given information about the normal to the plane.

If the plane is parallel to the line (or if the line is contained in the plane), the normal of the plane and the direction vector of the line will be perpendicular.

This implies that $\vec{n} \cdot \vec{m} = 0$.

If the line is not parallel to the plane, it must intersect at a single point.

Intersection of a Line and a Plane

Example

Determine whether the line $L : \vec{r} = (2, 0, -1) + t(1, 3, -2)$ intersects the plane $\pi : 4x - 2y - z + 5 = 0$.

Check if the normal to the plane and the direction vector of the line are perpendicular.

$$1(4) + 3(-2) - 2(-1) = 0$$

Therefore, the line is parallel to, or contained in, the plane.

Test the point on the line in the equation of the plane.

$$\begin{aligned} 4(2) - 2(0) - (-1) + 5 &= 14 \\ &\neq 0 \end{aligned}$$

The point is not in the plane, so the line and plane are parallel.

Intersection of a Line and a Plane

Example

Determine any points of intersection for the line $L : \vec{r} = (1, 0, -1) + t(-2, 3, 0)$ and the plane $\pi : 2x - 3y + z - 14 = 0$.

Check if the normal and direction vector are perpendicular.

$$2(-2) - 3(3) + 1(0) = -13$$

The normal and direction vector are not perpendicular, so there is a single point of intersection.

The parametric equations of the line are $x = 1 - 2t$, $y = 3t$ and $z = -1$.

Intersection of a Line and a Plane

The point of intersection will satisfy the equation of the plane for some value of the parameter t.

Substitute the parametric equations into the equation of the plane and solve for t.

$$\begin{aligned} 2(1 - 2t) - 3(3t) + (-1) - 14 &= 0 \\ -13t &= 13 \\ t &= -1 \end{aligned}$$

When $t = -1$, the line intersects the plane. This occurs at the point $(1 - 2(-1), 3(-1), -1) = (3, -3, -1)$.

Intersection of a Line and a Plane

If we are given the vector or parametric equations of a plane, we can find a normal to the plane using the cross product.

As before, if the dot product of the normal and the direction vector of the line are perpendicular, then the line and plane are parallel.

Intersection of a Line and a Plane

Example

Determine whether the line $L : \vec{r} = (1, 0, 3) + k(2, -1, 1)$ intersects the plane $\pi : \vec{r} = (1, 1, 0) + s(2, 0, -1) + t(1, 3, -2)$.

Find the normal to the plane using the cross product.

$$(2, 0, -1) \times (1, 3, -2) = (3, 3, 6)$$

Calculate the dot product of the normal and the direction vector of the line.

$$(2, -1, 1) \cdot (3, 3, 6) = 9$$

Therefore, the line and plane intersect.

Intersection of a Line and a Plane

Example

Determine any points of intersection for the line $L : \vec{r} = (1, 1, 3) + k(-1, 3, 0)$ and the plane $\pi : \vec{r} = (0, 2, -2) + s(1, 5, -2) + t(-1, -2, 3)$.

Use the cross product to determine a normal to the plane.

$$(1, 5, -2) \times (-1, -2, 3) = (11, -1, 3)$$

Check if the normal and the direction vector of the line are perpendicular.

$$(11, -1, 3) \cdot (-1, 3, 0) = -14$$

The line and plane intersect at a single point.

Intersection of a Line and a Plane

Convert the equation from vector to scalar form.

$$\begin{aligned} 11(0) - 1(2) + 3(-2) + D &= 0 \\ D &= 8 \\ 11x - y + 3z + 8 &= 0 \end{aligned}$$

Substitute the parametric equations into the scalar equation of the plane.

$$\begin{aligned} 11(1-k) - 1(1+3k) + 3(3) + 8 &= 0 \\ -14k &= -27 \\ k &= \tfrac{27}{14} \end{aligned}$$

When $t = \frac{27}{14}$, the line intersects the plane. This occurs at the point $\left(1 - \frac{27}{14}, 1 + 3\left(\frac{27}{14}\right), 3\right) = \left(-\frac{13}{14}, \frac{95}{14}, 3\right)$.

Questions?