MCV4U: Calculus & Vectors

Implicit Differentiation

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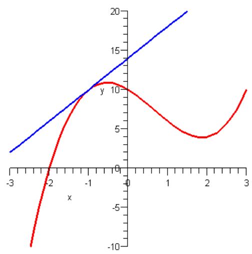

Quotient Rule

Recap

Determine the derivative of $f(x) = \dfrac{3x^2+5}{x-1}$.

Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x-1)(6x)-(3x^2+5)(1)}{(x-1)^2} \\ &= \frac{3x^2-6x-5}{(x-1)^2} \end{aligned}$$

Using the product and chain rules,

$$\begin{aligned} f(x) &= (3x^2+5)(x-1)^{-1} \\ f'(x) &= (6x)(x-1)^{-1}-(3x^2+5)(x-1)^{-2} \end{aligned}$$

which simplifies to the same after some work.

Implicit Differentiation

Consider the line $y = 3x+5$, whose derivative is $\frac{dy}{dx} = 3$.

As it is written, the equation of the line is *explicitly defined*, since it describes the line entirely in terms of x.

An equivalent equation for the line is $3x - y = -5$. This time, the equation is *implicitly defined*, since the function is defined in terms of both x and y.

While most functions that we have dealt with up to this point have been explicitly defined, some fuctions are impossible to express explicitly.

It is possible to determine the derivative of a function that is implicitly defined by using the basic derivate rules that we have established.

Implicit Differentiation

Example

Determine the derivative of the line described by $3x - y = -5$.

Take the derivative of both sides, noting that the right hand side is a constant.

$$\begin{aligned} \tfrac{d}{dx}(3x-y) &= \tfrac{d}{dx}(-5) \\ 3\tfrac{d}{dx}x - \tfrac{d}{dx}y &= 0 \end{aligned}$$

Note that $\frac{d}{dx}x = 1$ and $\frac{d}{dx}y = \frac{dy}{dx}$, so

$$3 - \tfrac{dy}{dx} = 0$$

Implicit Differentiation

Isolate $\frac{dy}{dx}$, which is the derivative that we are looking for.

$$\tfrac{dy}{dx} = 3$$

This is the same answer that we receive when differentiating $y = 3x+5$.

In this instance, implicit differentiation is more work than necessary, but there are other functions for which it is a time-saver, or where it is the only option.

Implicit Differentiation

In the previous example, setting $\frac{d}{dx}y = \frac{dy}{dx}$ obscured the fact that the chain rule was being used.

Consider the process of differentiating $y = x^2$.

From the power rule, we know that $\frac{dy}{dx} = 2x$.

Alternatively, consider y as a composite function, where the inner function is $u = x$ and the outer function is $y = u^2$.

Then $\frac{du}{dx} = 1$ and $\frac{dy}{du} = 2u$.

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 1 = 2x$.

Note that when calculating $\frac{dy}{dx}$, we find the derivative of y with respect to an "intermediate" variable u $\left(\frac{dy}{du}\right)$, then multiply it by its derivative with respect to x $\left(\frac{du}{dx}\right)$.

Implicit Differentiation

Example

Determine the derivative of the circle described by $x^2 + y^2 = 25$.

It is possible to define a circle explicitly using two equations, one for the upper semicircle and one for the lower.

In this case, the equations would be $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$.

Differentiating the first equation using the chain rule results in $\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$, or $\frac{dy}{dx} = -\frac{x}{\sqrt{25 - x^2}}$.

We could repeat the process for the second equation to obtain $\frac{dy}{dx} = \frac{x}{\sqrt{25 - x^2}}$.

Implicit Differentiation

Alternatively, we can implicitly differentiate $x^2 + y^2 = 25$ instead.

$$\begin{aligned} \tfrac{d}{dx}(x^2 + y^2) &= \tfrac{d}{dx}25 \\ \tfrac{d}{dx}x^2 + \tfrac{d}{dx}y^2 &= 0 \end{aligned}$$

At this point, $\frac{d}{dx}x^2$ is simply $2x$, since we are differentiating a function of x with respect to x.

$\frac{d}{dx}y^2$, however, is a function of y. So to differentiate this term, use the chain rule.

$$2x + 2y\tfrac{dy}{dx} = 0$$

Implicit Differentiation

As before, isolate $\frac{dy}{dx}$ to obtain the derivative.

$$\begin{aligned} \tfrac{dy}{dx} &= -\frac{2x}{2y} \\ &= -\frac{x}{y} \end{aligned}$$

Notice that in this case, the derivative is expressed in terms of both x and y.

Therefore, if we wished evaluate the derivative of the function, we would need to substitute values of x and y, rather than x alone.

Also notice, however, that since $y = \pm\sqrt{25 - x^2}$, then $\frac{dy}{dx} = \pm\frac{x}{\sqrt{25 - x^2}}$, which is the same result obtained earlier.

Implicit Differentiation

Example

Determine the derivative of the hyperbola described by $3x^2 - 2y^2 = 4$.

While possible to isolate y, implicit differentiation is faster.

$$\begin{aligned} \tfrac{d}{dx}(3x^2 - 2y^2) &= \tfrac{d}{dx}4 \\ 6x - 4y\tfrac{dy}{dx} &= 0 \\ \tfrac{dy}{dx} &= \frac{6x}{4y} \\ &= \frac{3x}{2y} \end{aligned}$$

Implicit Differentiation

Example

Determine the slope(s) of any tangent(s) to the ellipse described by $2x^2 + 3xy + 5y^2 = 4$ when $x = 1$.

It is not possible to isolate y in this relation, so explicit differentiation is not an option.

$$\begin{aligned} \tfrac{d}{dx}(2x^2 + 3xy + 5y^2) &= \tfrac{d}{dx}4 \\ 2\tfrac{d}{dx}x^2 + 3\tfrac{d}{dx}xy + 5\tfrac{d}{dx}y^2 &= 0 \end{aligned}$$

The first and third terms are not a problem, since we have dealt with them before.

$$4x + 3\tfrac{d}{dx}(xy) + 10y\tfrac{dy}{dx} = 0$$

Implicit Differentiation

The second term is the product of x and y, so we must use the product rule here.

$$\begin{aligned} 4x + 3\tfrac{d}{dx}(xy) + 10y\tfrac{dy}{dx} &= 0 \\ 4x + 3\left((1)(y) + (x)\left(\tfrac{dy}{dx}\right)\right) + 10y\tfrac{dy}{dx} &= 0 \\ 4x + 3y + 3x\tfrac{dy}{dx} + 10y\tfrac{dy}{dx} &= 0 \end{aligned}$$

To isolate $\frac{dy}{dx}$, we need to common factor.

$$\begin{aligned} 3x\tfrac{dy}{dx} + 10y\tfrac{dy}{dx} &= -(4x + 3y) \\ \tfrac{dy}{dx} &= -\frac{4x + 3y}{3x + 10y} \end{aligned}$$

Implicit Differentiation

Now that we have an expression for the derivative, we need to determine any points of tangency.

$$2(1)^2 + 3(1)y + 5y^2 = 4$$
$$5y^2 + 3y - 2 = 0$$
$$(y+1)(5y-2) = 0$$

Thus, two points of tangency are $(1, -1)$ and $\left(1, \frac{2}{5}\right)$.

Substitute these values into the derivative to find the slopes.

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = -\frac{4(1)+3(-1)}{3(1)+10(-1)} = \frac{1}{7} \qquad \left.\frac{dy}{dx}\right|_{\left(1,\frac{2}{5}\right)} = -\frac{4(1)+3\left(\frac{2}{5}\right)}{3(1)+10\left(\frac{2}{5}\right)} = -\frac{26}{35}$$

Questions?