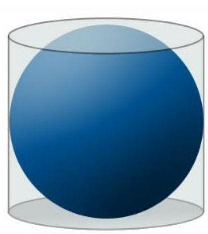
MCV4U: Calculus & Vectors

L'Hôpital's Rule

J. Garvin



L'Hôpital's Rule

Earlier in the course, we evaluated limits in indeterminate forms by factoring, using the conjugate, etc.

An alternative uses a technique called *l'Hôpital's Rule*.

L'Hôpital's Rule for Limits

If $\lim_{x\to a} f(x)$:

- can be written in the form $\lim_{x\to a} f(x) = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)}$, where both $g(x)$ and $h(x)$ are differentiable, and
- if $\lim_{x\to a} f(x) = \frac{0}{0}$ or $\lim_{x\to a} f(x) = \pm\frac{\infty}{\infty}$,

then $\lim_{x\to a} f(x) = \lim_{x\to a} \frac{g'(x)}{h'(x)}$.

L'Hôpital's Rule

Example

Determine $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$.

While we can solve this using factoring, l'Hôpital's rule applies since we obtain $0/0$.

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{2x+1}{1}$$

With no restrictions in the denominator, we can evaluate.

$$\begin{aligned}\lim_{x\to 2} \frac{2x+1}{1} &= \frac{2\lim_{x\to 2} x + \lim_{x\to 2} 1}{\lim_{x\to 2} 1} \\ &= \frac{2(2)+1}{1} \\ &= 5\end{aligned}$$

L'Hôpital's Rule

Example

Determine $\lim_{x\to 0} \frac{\sin x}{x}$.

We covered this limit earlier when developing the derivative of $f(x) = \sin x$, using a geometric argument to suggest it was true. Since we obtain $0/0$, use l'Hôpital's rule instead.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{\cos x}{1} \\ &= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} 1} \\ &= \tfrac{1}{1} \\ &= 1\end{aligned}$$

L'Hôpital's Rule

Example

Determine $\lim_{x\to\infty} \frac{2^x}{x^2-7}$.

Since this is a case of ∞/∞, l'Hôpital's rule applies.

$$\lim_{x\to\infty} \frac{2^x}{x^2-7} = \lim_{x\to\infty} \frac{2^x \ln 2}{2x}$$

This is still ∞/∞, so use l'Hôpital's rule again.

$$\lim_{x\to\infty} \frac{2^x \ln 2}{2x} = \lim_{x\to\infty} \frac{2^x(\ln 2)^2}{2}$$

Since $2^x(\ln 2)^2 \to \infty$ as $x \to \infty$, $\lim_{x\to\infty} \frac{2^x}{x^2-7}$ does not exist. The function grows without bound.

L'Hôpital's Rule

Example

Determine $\lim_{x\to 5} \frac{x^2+3}{x-4}$.

Using L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to 5} \frac{x^2+3}{x-4} &= \lim_{x\to 5} \tfrac{2x}{1} \\ &= 10\end{aligned}$$

This is incorrect, as shown using properties of limits.

$$\begin{aligned}\lim_{x\to 5} \frac{x^2+3}{x-4} &= \frac{\left[\lim_{x\to 5} x\right]^2 + \lim_{x\to 5} 3}{\lim_{x\to 5} x - \lim_{x\to 5} 4} \\ &= 28\end{aligned}$$

L'Hôpital's Rule

It is important to test that $\lim_{x\to a} = \frac{0}{0}$ or $\lim_{x\to a} = \pm\frac{\infty}{\infty}$.

The last example illustrates that l'Hôpital's rule can only be used if it meets either of these criteria.

It is also necessary to express the limit as a rational function. This may require some rewriting involving reciprocals.

L'Hôpital's Rule

Example

Determine $\lim_{x\to\infty} xe^{-x}$.

Rewrite the limit, using the fact that $b^{-k} = \frac{1}{b^k}$.

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$$

This gives us an ∞/∞ case, so we can use l'Hôpital's rule.

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x}$$

As $x \to \infty$, $e^x \to \infty$ as well. Therefore, since $\lim_{x\to\infty} \frac{1}{e^x} = 0$, $\lim_{x\to\infty} xe^{-x} = 0$.

L'Hôpital's Rule

Example

Determine $\lim_{x\to 0^+} \sqrt{x}\ln x$.

Rewrite the limit by multiplying by the reciprocal of $\sqrt{x}$.

$$\begin{aligned}\lim_{x\to 0^+} \sqrt{x}\ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}}\\ &= \lim_{x\to 0^+} \frac{1/x}{-1/(2x^{3/2})}\\ &= \lim_{x\to 0^+} (-2\sqrt{x})\end{aligned}$$

As $x \to 0^+$, $-2\sqrt{x} \to 0$. Therefore, since $\lim_{x\to 0^+} (-2\sqrt{x}) = 0$, $\lim_{x\to 0^+} \sqrt{x}\ln x = 0$.

Questions?