MCV4U: Calculus & Vectors

Distance From a Point To a Line/Plane

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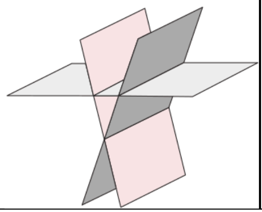


Distance From a Point to a Line

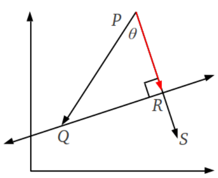
Visualize a line in two-space, and point $P(x_p, y_p)$ off the line.

If the line has equation $Ax + By + C = 0$, then a normal to the line is $\vec{n} = (A, B)$.

Construct vector $\vec{PS}$, parallel to $\vec{n}$, that intersects the line at point $R(x_r, y_r)$.

Let point $Q(x_q, y_q)$ be a point on the line, and construct $\vec{PQ} = (x_q - x_p, y_q - y_p)$ such that it makes an angle of θ with $\vec{PS}$.

Distance From a Point to a Line



To find the distance between point P and the line, we find the scalar projection of $\vec{PQ}$ onto $\vec{PS}$.

Since $\vec{PS}$ is parallel to $\vec{n}$, this is the same as finding the scalar projection of $\vec{PQ}$ onto $\vec{n}$.

The scalar projection will have a value equal to the magnitude of $\vec{PR}$.

Distance From a Point to a Line

$$\begin{aligned}|\text{proj}_{\vec{n}}\vec{PQ}| &= \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|} \\ |\vec{PR}| &= \frac{|(x_q - x_p, y_q - y_p) \cdot (A, B)|}{\sqrt{A^2 + B^2}} \\ &= \frac{|Ax_q + By_q - Ax_p - By_p|}{\sqrt{A^2 + B^2}}\end{aligned}$$

Since $Q(x_q, y_q)$ is on the line, $Ax_q + By_q + C = 0$, or $Ax_q + By_q = -C$.

Distance From a Point to a Line

$$\begin{aligned}|\vec{PR}| &= \frac{|-C - Ax_p - By_p|}{\sqrt{A^2 + B^2}} \\ &= \frac{|-(Ax_p + By_p + C)|}{\sqrt{A^2 + B^2}}\end{aligned}$$

The absolute value of the numerator means we can omit the negative sign.

Distance From a Point to a Line In Two-Space

The distance between a point, $P(x_p, y_p)$, and a line, $Ax + By + C = 0$, is given by $d = \dfrac{|Ax_p + By_p + C|}{\sqrt{A^2 + B^2}}$.

Distance From a Point to a Line

Example

Determine the distance from the point $P(3, 5)$ to the line $2x - y + 7 = 0$.

$$\begin{aligned}d &= \frac{|2(3) - (5) + 7|}{\sqrt{2^2 + (-1)^2}} \\ &= \frac{8}{\sqrt{5}} \\ &\approx 3.6 \text{ units}\end{aligned}$$

Distance From a Point to a Line

Example

Determine the distance from the point $P(-1, 3)$ to the line $\vec{r} = (6, 2) + t(4, -2)$.

Begin by converting the vector equation to scalar form, with normal vector $(2, 4)$ or, even better, $(1, 2)$.

$$\begin{aligned} x + 2y + C &= 0 \\ 6 + 2(2) + C &= 0 \\ C &= -10 \\ x + 2y - 10 &= 0 \end{aligned}$$

Distance From a Point to a Line

Now the distance can be calculated using the previous formula.

$$\begin{aligned} d &= \frac{|1(-1) + 2(3) - 10|}{\sqrt{1^2 + 2^2}} \\ &= \frac{5}{\sqrt{5}} \\ &\approx 2.2 \text{ units} \end{aligned}$$

Distance Between Parallel Lines

Example

Determine the distance between the lines $2x - 3y + 4 = 0$ and $2x - 3y + 9 = 0$.

The two lines are parallel because they have the same normal vector, $(2, -3)$.

A point on the first line is $(-2, 0)$, found by setting $y = 0$.

Use the formula to find the distance between $(-2, 0)$ and $2x - 3y + 9 = 0$.

$$\begin{aligned} d &= \frac{|2(-2) - 3(0) + 9|}{\sqrt{2^2 + (-3)^2}} \\ &= \frac{5}{\sqrt{13}} \\ &\approx 1.4 \text{ units} \end{aligned}$$

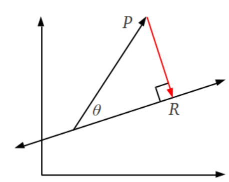
Distance From a Point to a Line

There is no scalar equation of a line in three-space, so we cannot use the same technique.

Again, visualize a line and a point $P(x_p, y_p, z_p)$ off of the line.

Let $Q(x_q, y_q, z_q)$ be a known point on the line, and construct $\vec{QP}$, which makes an angle of θ with the line.

Let $R(x_r, y_r, z_r)$ be a point in the line such that $\vec{PR}$ is perpendicular to the line.



Distance From a Point to a Line

In the diagram, $|\vec{PR}| = |\vec{QP}| \sin\theta$.

Using the cross product, $|\vec{m} \times \vec{QP}| = |\vec{m}||\vec{QP}| \sin\theta$.

Substituting the expression for $|\vec{PR}|$, we obtain $|\vec{m} \times \vec{QP}| = |\vec{m}||\vec{PR}|$.

Rearranging, we get $|\vec{PR}| = \frac{|\vec{m} \times \vec{QP}|}{|\vec{m}|}$.

Distance From a Point to a Line In Three-Space

The distance between a point, $P(x_p, y_p, z_p)$, and a line, $\vec{r} = (x_q, y_q, z_q) + t(x_m, y_m, z_m)$, is given by $d = \dfrac{|\vec{m} \times \vec{QP}|}{|\vec{m}|}$.

Distance From a Point to a Line

Example

Determine the distance from $P(-1, 1, 6)$ to $\vec{r} = (1, 2, -1) + t(0, 1, 1)$.

$$\begin{aligned} \vec{QP} &= (-2, -1, 7) \\ d &= \frac{|(0, 1, 1) \times (-2, -1, 7)|}{|(0, 1, 1)|} \\ &= \frac{|(8, -2, 2)|}{|(0, 1, 1)|} \\ &= \frac{\sqrt{8^2 + (-2)^2 + 2^2}}{\sqrt{0^2 + 1^2 + 1^2}} \\ &= 6 \text{ units} \end{aligned}$$

Distance From a Point To a Plane

A formula for the distance from a point to a plane is similar to that for the distance from a point to a line.

Since the derivation is almost identical, it is omitted here.

Distance From a Point To a Plane

The distance between a point, $P(x_p, y_p, z_p)$, and a plane, $Ax + By + Cz + D = 0$, is given by $d = \dfrac{|Ax_p + By_p + Cz_p + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Distance From a Point To a Plane

Example

Determine the distance from $P(1, 2, 3)$ to the plane $3x - y - 2z + 1 = 0$.

$$\begin{aligned} d &= \frac{|3(1) - 1(2) - 2(3) + 1|}{\sqrt{3^2 + (-1)^2 + (-2)^2}} \\ &= \frac{4}{\sqrt{14}} \\ &\approx 1.07 \text{ units} \end{aligned}$$

Distance Between Skew Lines

Example

Determine the distance between the skew lines $\vec{r_1} = (1, 0, -2) + s(3, 1, 1)$ and $\vec{r_2} = (1, 1, -3) + t(-2, 1, 0)$.

To find the distance, first determine an equation of a plane that contains $\vec{r_1}$ and is parallel to $\vec{r_2}$.

This means that $\vec{m_1} = (3, 1, 1)$ and $\vec{m_2} = (-2, 1, 0)$ are direction vectors for π_1.

A normal to the plane, $\vec{n_1}$, is $(3, 1, 1) \times (-2, 1, 0) = (1, 2, -5)$.

Therefore, π_1 has a scalar equation $x + 2y - 5z + D = 0$.

Substituting point $(1, 0, -2)$ from $\vec{r_1}$ gives a scalar equation of $\pi_1 : x + 2y - 5z - 11 = 0$.

Distance Between Skew Lines

The problem is now the same as finding the distance from a point on $\vec{r_2}$ to π_1.

Use the formula with the point $(1, 1, -3)$.

$$\begin{aligned} d &= \frac{|1(1) + 2(1) - 5(-3) - 11|}{\sqrt{1^2 + 2^2 + (-5)^2}} \\ &= \frac{7}{\sqrt{30}} \\ &\approx 1.28 \text{ units} \end{aligned}$$

Questions?