MCV4U: Calculus & Vectors

# Derivatives of Sinusoidal Functions

(Sine and Cosine)

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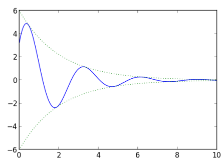


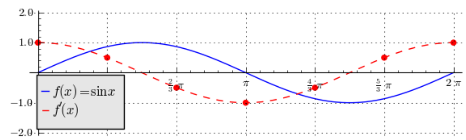




## Derivative of $f(x) = \sin x$

While we have dealt with derivatives of polynomial, radical and reciprocal functions, we have not yet dealt with the derivatives of sinusoidal functions like $f(x) = \sin x$.

We might calculate the rate of change every $\frac{\pi}{3}$ radians and compare it to the graph of sine.




From the graph, it appears that the rate of change of the sine function is the cosine function.





## Derivative of $f(x) = \sin x$

To establish the derivative of the sine function, use the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Use the angle sum identity for sine.

$$f'(x) = \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

Common factor $\sin x$.

$$f'(x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h}$$





## Derivative of $f(x) = \sin x$

Use the multiplicative property of limits.

$$f'(x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h \cos x}{h}$$

Since we are concerned with the limit of $h$, both $\sin x$ and $\cos x$ can be treated like constants outside of the limits.

$$f'(x) = \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

At this point, we need to establish values for the two limits above.

Proofs of these limits generally rely on the Squeeze Theorem and some trigonometric inequalities, but some informal arguments should convince us of their values.

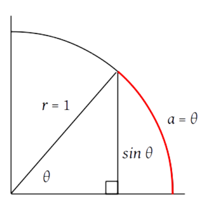




## Derivative of $f(x) = \sin x$

Recall that in radian measure, arc length of a circle is given by $a = r\theta$, where $\theta$ is the angle subtending the arc and $r$ is the radius of the circle.

Consider the unit circle below. When $r = 1$, $a = \theta$. The height of a right triangle inside of the sector has a height of $\sin \theta$.

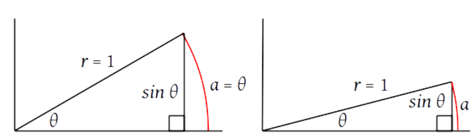







## Derivative of $f(x) = \sin x$

Now consider what happens as $\theta \to 0$.



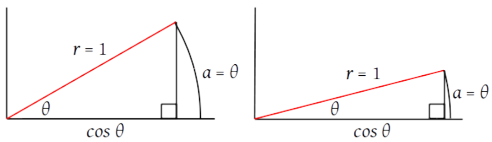

The length of the arc, $\theta$, and the height of the triangle, $\sin \theta$, become closer in value. If $\theta$ was infinitesimally small, then $\theta$ and $\sin \theta$ would essentially have the same value.

Their ratio, $\frac{\sin \theta}{\theta}$, would be 1. Therefore, $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

## Derivative of $f(x) = \sin x$

A similar argument can be used for $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$.

Consider what happens to the horizontal component of a right triangle, $\cos\theta$, as $\theta \to 0$.




If $\theta$ was infinitesimally small, then $\cos\theta$ would essentially have the same length as the radius, 1.

Therefore, $\cos\theta - 1 = 1 - 1 = 0$, and $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$.





## Derivative of $f(x) = \sin x$

Now return to the previous definition of the derivative of $f(x) = \sin x$ and substitute these values.

$$\begin{aligned} f'(x) &= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x \end{aligned}$$

This confirms our graph earlier.

**Derivative of the Sine Function**

If $f(x) = \sin x$, then $f'(x) = \cos x$.





## Derivative of $f(x) = \cos x$

Use the same process for the derivative of $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x \end{aligned}$$

**Derivative of the Cosine Function**

If $f(x) = \cos x$, then $f'(x) = -\sin x$.





## Derivatives Involving Sinusoidal Functions

We can use the derivative rules developed earlier to find the derivatives of functions involving either sine or cosine.

**Example**

Determine the derivative of $f(x) = 10\cos x + 4$.

Since 10 is a constant multiple, and 4 is a constant, $f'(x) = -10\sin x$.

**Example**

Determine the derivative of $f(x) = \sin x - \cos x$.

Using the difference rule for derivatives, $f'(x) = \cos x - (-\sin x) = \cos x + \sin x$.





## Derivatives Involving Sinusoidal Functions

**Example**

Determine the derivative of $y = \cos(2x^3 + 5x)$.

Use the chain rule, where the inner function is $u = 2x^3 + 5x$ and the outer function is $y = \cos u$.

$$\frac{dy}{dx} = -\sin(2x^3 + 5x)(6x^2 + 5)$$

**Example**

Determine the derivative of $y = 7\sin^3 x + 2\cos^2 x$.

Use the chain rule and the fact that $sin^n x = (\sin x)^n$.

$$\begin{aligned} \frac{dy}{dx} &= 21(\sin x)^2(\cos x) + 4(\cos x)(-\sin x) \\ &= 21\sin^2 x \cos x - 4\sin x \cos x \end{aligned}$$





## Derivatives Involving Sinusoidal Functions

**Example**

Determine the derivative of $y = 2\sin x \cos x$.

Use the product rule to differentiate.

$$\begin{aligned} \frac{dy}{dx} &= 2(\cos x \cos x - \sin x \sin x) \\ &= 2(\cos^2 x - \sin^2 x) \\ &= 2\cos 2x \end{aligned}$$

An alternative solution uses the chain rule and the identity $y = 2\sin x \cos x = \sin 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \cos(2x)(2) \\ &= 2\cos 2x \end{aligned}$$

## Derivatives Involving Sinusoidal Functions

**Example**

Determine the slope of the tangent to $f(x) = 2x\cos x$ when $x = \frac{5\pi}{3}$.

Use the product rule to differentiate.

$$\begin{aligned} f'(x) &= 2\cos x + 2x(-\sin x) \\ &= 2(\cos x - x\sin x) \end{aligned}$$

Find $f'\left(\frac{5\pi}{3}\right)$ for the slope of the tangent.

$$\begin{aligned} f'\left(\tfrac{5\pi}{3}\right) &= 2\left(\tfrac{1}{2} - \tfrac{5\pi}{3}\left(-\tfrac{\sqrt{3}}{2}\right)\right) \\ &= 1 + \tfrac{5\sqrt{3}\pi}{3} \end{aligned}$$





## Derivatives Involving Sinusoidal Functions

**Example**

Find any values of $x$ for which the tangent to $y = 3\sin^2 x$, on the domain $[0, 2\pi]$, has a slope of $\frac{3}{2}$.

Using the chain rule, the derivative is $\frac{dy}{dx} = 6\sin x\cos x$, or $\frac{dy}{dx} = 3\sin 2x$ after applying the double-angle formula.

Set $\frac{dy}{dx} = \frac{3}{2}$ and solve for $2x$.

$$\begin{aligned} 3\sin 2x &= \tfrac{3}{2} \\ \sin 2x &= \tfrac{1}{2} \\ 2x &= \sin^{-1}\left(\tfrac{1}{2}\right) \\ 2x &= \tfrac{\pi}{6} \end{aligned}$$





## Derivatives Involving Sinusoidal Functions

Since $\sin 2x$ has a period of $\pi$, two cycles will be completed on the interval $[0, 2\pi]$. Thus, there should be four values of $x$ for which this equation is true.

The first two can be found by using $2x$ as the reference angle.

$$\begin{aligned} 2x &= \tfrac{\pi}{6} \\ x &= \tfrac{\pi}{12} \end{aligned}$$

$$\begin{aligned} \pi - 2x &= \tfrac{\pi}{6} \\ x &= \frac{\pi - \frac{\pi}{6}}{2} \\ &= \tfrac{5\pi}{12} \end{aligned}$$

Find the other two by adding the period, $\pi$, to each value.

$$\begin{aligned} x &= \pi + \tfrac{\pi}{12} \\ &= \tfrac{13\pi}{12} \end{aligned}$$

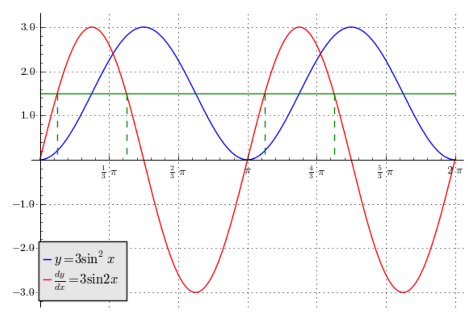
$$\begin{aligned} x &= \pi + \tfrac{5\pi}{12} \\ &= \tfrac{17\pi}{12} \end{aligned}$$





## Derivatives Involving Sinusoidal Functions

A graph of $y$ and $\frac{dy}{dx}$ is below.








## Questions?